New Keynesian Model
Nominal Wage Rigidities à la Calvo

Damian Romero*

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1 One-sector model with representative agent

1.1 Optimal wage

The problem of the union is

$$\max_{W_t^*} \mathbb{E}_t \sum_{k=0}^{\infty} (\theta_w \beta)^k U(C_{t+k|t}, N_{t+k|t}),$$

subject to

$$N_{t+k|t} = \left(\frac{W_t^*}{W_{t+k}} \right)^{-\varepsilon_w} N_{t+k} \tag{1}$$

$$P_{t+k} C_{t+k|t} + Q_{t+k} B_{t+k} = B_{t+k-1} + W_t^* N_{t+k|t} + D_{t+k} + T_{t+k}. \tag{2}$$

Even though we are interested in determining the optimal nominal wage W_t^*, we can re-write the constraints (1) and (2) such that every nominal price is in terms of the numeraire: the price of consumption, P_t. These restrictions read as

$$\begin{aligned} N_{t+k|t} &= \left(\frac{W_t^*}{P_t} \frac{P_t}{P_{t+k}} \frac{P_{t+k}}{W_{t+k}} \right)^{-\varepsilon_w} N_{t+k} \\ &= \left(\frac{W_t^*}{P_t} \right)^{-\varepsilon_w} \left(\frac{P_{t+k}}{P_t} \frac{W_{t+k}}{P_{t+k}} \right)^{\varepsilon_w} N_{t+k} \end{aligned} \tag{3}$$

$$\begin{aligned} C_{t+k|t} &= \frac{B_{t+k-1} - Q_{t+k} B_{t+k} + W_t^* N_{t+k|t} + D_{t+k} + T_{t+k}}{P_{t+k|t}} \\ &= \frac{B_{t+k-1} - Q_{t+k} B_{t+k} + D_{t+k} + T_{t+k}}{P_{t+k|t}} + \frac{W_t^*}{P_t} \frac{P_t}{P_{t+k}} N_{t+k|t} \\ &= \frac{B_{t+k-1} - Q_{t+k} B_{t+k} + D_{t+k} + T_{t+k}}{P_{t+k|t}} + \frac{W_t^*}{P_t} \frac{P_t}{P_{t+k}} \left(\frac{W_t^*}{P_t} \right)^{-\varepsilon_w} \left(\frac{P_{t+k}}{P_t} \frac{W_{t+k}}{P_{t+k}} \right)^{\varepsilon_w} N_{t+k} \\ &= \frac{B_{t+k-1} - Q_{t+k} B_{t+k} + D_{t+k} + T_{t+k}}{P_{t+k|t}} + \left(\frac{W_t^*}{P_t} \right)^{1-\varepsilon_w} \left(\frac{P_{t+k}}{P_t} \right)^{\varepsilon_w - 1} \left(\frac{W_{t+k}}{P_{t+k}} \right)^{\varepsilon_w} N_{t+k}. \end{aligned} \tag{4}$$

*Email: dromeroc@bcentral.cl

Then, the optimization problem is unrestricted because we can replace for consumption and labor from equations (3) and (4).
The first order condition is

$$\mathbb{E}_t \sum_{k=0}^{\infty} (\theta_w \beta)^k \Bigg\{ U_{C,t+k}(1-\varepsilon_w) \left(\frac{W_t^*}{P_t}\right)^{-\varepsilon_w} \frac{1}{P_t} \left(\frac{P_{t+k}}{P_t}\right)^{\varepsilon_w - 1} \left(\frac{W_{t+k}}{P_{t+k}}\right)^{\varepsilon_w} N_{t+k} \\ - U_{N,t+k}\varepsilon_w \left(\frac{W_t^*}{P_t}\right)^{-\varepsilon_w - 1} \frac{1}{P_t} \left(\frac{P_{t+k}}{P_t}\frac{W_{t+k}}{P_{t+k}}\right)^{\varepsilon_w} N_{t+k} \Bigg\} = 0$$

$$\mathbb{E}_t \sum_{k=0}^{\infty} (\theta_w \beta)^k \Bigg\{ U_{C,t+k} \left(\frac{W_t^*}{P_t}\right) \left(\frac{P_{t+k}}{P_t}\right)^{\varepsilon_w - 1} \left(\frac{W_{t+k}}{P_{t+k}}\right)^{\varepsilon_w} N_{t+k} + \left(\frac{\varepsilon_w}{\varepsilon_w - 1}\right) U_{N,t+k} \left(\frac{P_{t+k}}{P_t}\frac{W_{t+k}}{P_{t+k}}\right)^{\varepsilon_w} N_{t+k} \Bigg\} = 0$$

$$\mathbb{E}_t \sum_{k=0}^{\infty} (\theta_w \beta)^k U_{C,t+k} \Bigg\{ \left(\frac{W_t^*}{P_t}\right) \left(\frac{P_{t+k}}{P_t}\right)^{\varepsilon_w - 1} \left(\frac{W_{t+k}}{P_{t+k}}\right)^{\varepsilon_w} N_{t+k} - \left(\frac{\varepsilon_w}{\varepsilon_w - 1}\right) \left(\frac{P_{t+k}}{P_t}\frac{W_{t+k}}{P_{t+k}}\right)^{\varepsilon_w} N_{t+k} MRS_{t+k} \Bigg\} = 0,$$

with $MRS_{t+k} = U_{N,t+k}/U_{C,t+k}$.
This can be also written as

$$\left(\frac{W_t^*}{P_t}\right) \underbrace{\mathbb{E}_t \sum_{k=0}^{\infty} (\theta_w \beta)^k U_{C,t+k} \Bigg\{ \left(\frac{P_{t+k}}{P_t}\right)^{\varepsilon_w - 1} \left(\frac{W_{t+k}}{P_{t+k}}\right)^{\varepsilon_w} N_{t+k} \Bigg\}}_{F_t^w} = \\ \left(\frac{\varepsilon_w}{\varepsilon_w - 1}\right) \underbrace{\mathbb{E}_t \sum_{k=0}^{\infty} (\theta_w \beta)^k U_{C,t+k} \Bigg\{ \left(\frac{P_{t+k}}{P_t}\frac{W_{t+k}}{P_{t+k}}\right)^{\varepsilon_w} N_{t+k} MRS_{t+k} \Bigg\}}_{S_t^w}.$$

Under $U(C,N) = \frac{C^{1-\sigma}-1}{1-\sigma} - \frac{N^{1+\varphi}}{1+\varphi}$ we have $U_C = C^{-\sigma}$, $U_N = N^{\varphi}$ and $MRS = C^{\sigma} N^{\varphi}$.
Working every expression we get

$$\begin{aligned} F_t^w &= \left(\frac{W_t}{P_t}\right)^{\varepsilon_w} N_t C_t^{-\sigma} + \theta_w \beta \mathbb{E}_t \sum_{k=0}^{\infty} (\theta_w \beta)^k U_{C,t+k+1} \Bigg\{ \left(\frac{P_{t+k+1}}{P_{t+1}}\frac{P_{t+1}}{P_t}\right)^{\varepsilon_w - 1} \left(\frac{W_{t+k+1}}{P_{t+k+1}}\right)^{\varepsilon_w} N_{t+k+1} \Bigg\} \\ &= \left(\frac{W_t}{P_t}\right)^{\varepsilon_w} N_t C_t^{-\sigma} + \theta_w \beta \mathbb{E}_t \left[\Pi_{t+1}^{\varepsilon_w - 1} F_{t+1}^w \right] \\ S_t^w &= \left(\frac{W_t}{P_t}\right)^{\varepsilon_w} N_t^{1+\varphi} + \theta_w \beta \mathbb{E}_t \sum_{k=0}^{\infty} (\theta_w \beta)^k U_{C,t+k+1} \Bigg\{ \left(\frac{P_{t+k+1}}{P_{t+1}}\frac{P_{t+1}}{P_t}\frac{W_{t+k+1}}{P_{t+k+1}}\right)^{\varepsilon_w} N_{t+k+1} MRS_{t+k+1} \Bigg\} \\ &= \left(\frac{W_t}{P_t}\right)^{\varepsilon_w} N_t^{1+\varphi} + \theta_w \beta \mathbb{E}_t \left[\Pi_{t+1}^{\varepsilon_w} S_{t+1}^w \right]. \end{aligned}$$

Then the optimal (real) wage is

$$\frac{W_t^*}{P_t} = \left(\frac{\varepsilon_w}{\varepsilon_w - 1}\right) \frac{S_t^w}{F_t^w}.$$

Note that when wages are fully flexible $S_t^w = \left(\frac{W_t}{P_t}\right)^{\varepsilon_w} N_t^{1+\varphi}$ and $F_t^w = \left(\frac{W_t}{P_t}\right)^{\varepsilon_w} N_t C_t^{-\sigma}$ so $\frac{W_t^*}{P_t} = \left(\frac{\varepsilon_w}{\varepsilon_w - 1}\right) C_t^{\sigma} N_t^{\varphi} = \left(\frac{\varepsilon_w}{\varepsilon_w - 1}\right) MRS_t$.

1.2 Wage evolution

Real wages ($w_t \equiv W_t / P_t$) evolve as $w_t = w_{t-1} \frac{\Pi_{w,t}}{\Pi_t}$.

Given the Calvo assumption, wages evolve as

$$
\begin{aligned}
W_t &= \left[\theta_w W_{t-1}^{1-\varepsilon_w} + (1-\theta_w)(W_t^*)^{1-\varepsilon_w} \right]^{\frac{1}{1-\varepsilon_w}} \\
1 &= \theta_w \left(\frac{W_{t-1}}{W_t} \right)^{1-\varepsilon_w} + (1-\theta_w) \left(\frac{W_t^*}{W_t} \right)^{1-\varepsilon_w} \\
1 &= \theta_w (\Pi_{w,t+1})^{\varepsilon_w - 1} + (1-\theta_w) \left(\frac{W_t^*}{P_t} \frac{P_t}{W_t} \right)^{1-\varepsilon_w} \\
1 &= \theta_w (\Pi_{w,t+1})^{\varepsilon_w - 1} + (1-\theta_w)(w_t^*)^{1-\varepsilon_w} w_t^{\varepsilon_w - 1}.
\end{aligned}
$$

Finally, the price dispersion evolves as

$$
\begin{aligned}
\Delta_{w,t} &= \int_0^1 \left(\frac{W_t(i)}{W_t} \right)^{-\varepsilon_w} di = \int_0^{\theta_w} \left(\frac{W_{t-1}(i)}{W_t} \right)^{-\varepsilon_w} di + \int_{\theta_w}^1 \left(\frac{W_t^*}{W_t} \right)^{-\varepsilon_w} di \\
&= \int_0^{\theta_w} \left(\frac{W_{t-1}(i)}{W_{t-1}} \frac{W_{t-1}}{W_t} \right)^{-\varepsilon_w} di + (1-\theta_w) \left(\frac{W_t^*}{P_t} \frac{P_t}{W_t} \right)^{-\varepsilon_w} \\
&= \theta_w \Pi_{w,t}^{\varepsilon_w} \Delta_{w,t-1} + (1-\theta_w)(w_t^*)^{-\varepsilon_w} w_t^{\varepsilon_w}.
\end{aligned}
$$