Derivation of the unit cost function

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March, 2018

In this note I compute the unit cost function for a Cobb-Douglas technology. The production function is composed by the usage of labor, capital and intermediate inputs, but is general enough to consider any kind of input. In general, the model will consider constant returns to scale.

1 Homogenous intermediate use

Consider the following production function

$$y_{jt} = e^{z_{jt}} l_{jt}^{\alpha_j^l} k_{jt}^{\alpha_j^k} \prod_{i=1}^{N} x_{ijt}^{a_{ij}}, \tag{1}$$

where output of sector j is produced using labor (l), capital (k) and intermediate inputs (x) from the rest of the economy. The term z is the productivity shock. A more general version of (1) would be

$$y_t = e^{z_t} \prod_{i=1}^{N} x_{it}^{a_i},$$

where x_i is an input for production and a_i measures the usage intensity.

A usual assumption is constant returns to scale ($\alpha_j^l + \alpha_j^k + \sum_{i=1}^{N} a_{ij} = 1$.)

The unit cost function tells you the cost of producing one unit of output. This is, it corresponds to the value function of the following problem[1]

$$\arg\min w_t l_{jt} + r_t^k k_{jt} + \sum_{i=1}^{n} p_{it} x_{it} \qquad \text{subject to} \qquad 1 = e^{z_{jt}} l_{jt}^{\alpha_j^l} k_{jt}^{\alpha_j^k} \prod_{i=1}^{N} x_{ijt}^{a_{ij}} \tag{2}$$

Denoting as λ_j the Lagrange multiplier of the problem, the first order conditions are

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[1] As you would notice later, this function is linear in the amount of output, so the cost of producing y units of output equals the unit cost function times y.

$$l_{jt} : w_t = \alpha_j^l \frac{y_{jt}}{l_{jt}} \lambda_j \tag{3}$$

$$k_{jt} : r_t^k = \alpha_j^k \frac{y_{jt}}{k_{jt}} \lambda_j \tag{4}$$

$$x_{ijt} : p_{it} = a_{ij} \frac{y_{jt}}{x_{ijt}} \lambda_j \tag{5}$$

From (3) we have $\lambda_j = (w_t l_{jt})/(a_{ij} y_{jt})$. Replacing in (4) and (5) we obtain $k_{jt} = [(w_t a_{ij})/(p_{it} a_j^l)] l_{jt}$ and $x_{ijt} = [(w_t a_{ij})/(p_{it} a_j^l)] l_{jt}$. Therefore, replacing in the constraint, we obtain that the amount of labor would be

$$\begin{aligned} l_{jt} &= e^{-z_{jt}} \left(\frac{r_t^k}{w_t} \frac{a_j^l}{a_{j^k}} \right)^{a_j^k} \left(\frac{a_j^l}{w_t} \right)^{\sum_{i=1}^N a_{ij}} \prod_{i=1}^N \left(\frac{p_{it}}{a_{ij}} \right)^{a_{ij}} \\ &= e^{-z_{jt}} \left(\frac{r_t^k}{a_{j^k}} \right)^{a_j^k} \left(\frac{w_t}{a_j^l} \right)^{a_j^l - 1} \prod_{i=1}^N \left(\frac{p_{it}}{a_{ij}} \right)^{a_{ij}}. \end{aligned}$$

Capital and intermediates are

$$\begin{aligned} k_{jt} &= e^{-z_{jt}} \left(\frac{r_t^k}{a_{j^k}} \right)^{a_j^k - 1} \left(\frac{w_t}{a_j^l} \right)^{a_j^l} \prod_{i=1}^N \left(\frac{p_{it}}{a_{ij}} \right)^{a_{ij}} \\ x_{ijt} &= e^{-z_{jt}} \left(\frac{r_t^k}{a_{j^k}} \right)^{a_j^k} \left(\frac{w_t}{a_j^l} \right)^{a_j^l} \left(\frac{a_{ij}}{p_{it}} \right) \prod_{i=1}^N \left(\frac{p_{it}}{a_{ij}} \right)^{a_{ij}}. \end{aligned}$$

Finally, the value function (unit cost function is)

$$\begin{aligned} c(\mathbf{p}) = e^{-z_{jt}} \Bigg[& w_t \left(\frac{r_t^k}{a_{j^k}} \right)^{a_j^k} \left(\frac{w_t}{a_j^l} \right)^{a_j^l - 1} \prod_{i=1}^N \left(\frac{p_{it}}{a_{ij}} \right)^{a_{ij}} + r_t^k \left(\frac{r_t^k}{a_{j^k}} \right)^{a_j^k - 1} \left(\frac{w_t}{a_j^l} \right)^{a_j^l} \prod_{i=1}^N \left(\frac{p_{it}}{a_{ij}} \right)^{a_{ij}} \\ & + p_{it} \left(\frac{r_t^k}{a_{j^k}} \right)^{a_j^k} \left(\frac{w_t}{a_j^l} \right)^{a_j^l} \left(\frac{a_{ij}}{p_{it}} \right) \prod_{i=1}^N \left(\frac{p_{it}}{a_{ij}} \right)^{a_{ij}} \Bigg] \\ & = B_j (w_t)^{a_j^l} (r_t^k)^{a_j^k} \prod_{i=1}^N (p_{it})^{a_{ij}} \end{aligned}$$

with $B_{jt} \equiv e^{-z_{jt}} (a_j^l)^{-a_j^l} (a_j^k)^{-a_j^k} \prod_{i=1}^N (a_{ij})^{-a_{ij}}$, and where $\mathbf{p}$ represents the vector of relevant prices for the problem of the firm.

A consequence of this model is, because of constant returns to scale, there are no profits, so $p_{jt} = c(\mathbf{p})$ in equilibrium.