New Keynesian Model
Optimal pricing under Calvo assumption

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November, 2020

This note summarizes the optimal pricing problem of a firm in the New Keynesian model (NK) under Calvo pricing. It assumes different setups (one sector model vs multisector), and different ways of specifying the way firms discount profits.

1 General formulation

As usual in the NK model with Calvo pricing, there is a probability $1-\theta$ that a firm can update its price in a given period, while with probability θ is cannot and must charge the aggregate or sectoral price of the last period.

The optimization problem of a monopolistically competitive firm z is

$$\max_{P_t^*(z)} \sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \{ Q_{t,t+k} [P_t^*(z) Y_{t+k|t}(z) - MC_{t+k} Y_{t+k|t}(z)] \} \quad \text{subject to} \quad Y_{t+k|t}(z) = \left(\frac{P_t^*(z)}{P_{t+k}} \right)^{-\varepsilon} Y_{t+k}. \quad (1)$$

Equation (1) indicates that a firm z maximizes the full stream of profits by choosing the optimal price $P_t^*(z)$ under the possibility that this price remains over time. $Q_{t,t+k}$ is the discount factor of the firm, that depending on the way we specify the problem, will take different forms. $Y_{t+k|t}(z)$ is the output produced by the firm in period $t+k$, conditional on the last reseted price in period t. This amount comes from the optimization of an aggregator or consumer that has CES preferences over the different varieties of the good. Therefore, P_t and Y_t denotes the aggregate prices and quantity produced, that can be interpreted as the aggregate in the overall economy (in a single sector model), or the aggregate sectoral variables (in a multisector model). One assumption in all these formulations is that the firm has constant returns to scale, so the marginal cost MC_t does not depend on the quantity produced.

The first order condition reads as

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left\{ Q_{t,t+k} \left[P_t^*(z) P_{t+k}^{\varepsilon} Y_{t+k} - \left(\frac{\varepsilon}{\varepsilon - 1} \right) MC_{t+k} P_{t+k}^{\varepsilon} Y_{t+k} \right] \right\} = 0. \quad (2)$$

In what follows, I study the solution of this optimal pricing problem under different configurations of the model.

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2 One sector economy with representative agent

2.1 Recursive formulation

Under this formulation, the stochastic discount factor is given by $Q_{t,t+k} = \beta^k \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+k}}$. Replacing the demand function and the stochastic discount factor in (2) we have

$$\sum_{k=0}^{\infty}(\theta\beta)^k \mathbb{E}_t \left\{ \left(\frac{C_{t+k}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+k}} \left[P_t^*(z) P_{t+k}^{\varepsilon} Y_{t+k} - \left(\frac{\varepsilon}{\varepsilon-1}\right) MC_{t+k} P_{t+k}^{\varepsilon} Y_{t+k} \right] \right\} = 0.$$

Dropping the elements that do not depend on k

$$\sum_{k=0}^{\infty}(\theta\beta)^k \mathbb{E}_t \left\{ C_{t+k}^{-\sigma} \frac{1}{P_{t+k}} \left[P_t^*(z) P_{t+k}^{\varepsilon} Y_{t+k} - \left(\frac{\varepsilon}{\varepsilon-1}\right) MC_{t+k} P_{t+k}^{\varepsilon} Y_{t+k} \right] \right\} = 0.$$

Dividing by P_t and by P_t^{ε} and defining the real marginal cost as $MC_t^r = MC_t/P_t$

$$\frac{P_t^*(z)}{P_t} \underbrace{\sum_{k=0}^{\infty}(\theta\beta)^k \mathbb{E}_t \left\{ C_{t+k}^{-\sigma} Y_{t+k} \left(\frac{P_{t+k}}{P_t}\right)^{\varepsilon-1} \right\}}_{F_t} = \left(\frac{\varepsilon}{\varepsilon-1}\right) \underbrace{\sum_{k=0}^{\infty}(\theta\beta)^k \mathbb{E}_t \left\{ C_{t+k}^{-\sigma} Y_{t+k} MC_{t+k}^r \left(\frac{P_{t+k}}{P_t}\right)^{\varepsilon} \right\}}_{S_t} \tag{3}$$

Now we want to find the recursive form of each expression. For F_t

$$\begin{aligned}
F_t &= C_t^{-\sigma} Y_t + \theta\beta \sum_{k=0}^{\infty}(\theta\beta)^k \mathbb{E}_t \left\{ C_{t+k+1}^{-\sigma} Y_{t+k+1} \left(\frac{P_{t+k+1}}{P_t}\right)^{\varepsilon-1} \left(\frac{P_{t+1}}{P_{t+1}}\right)^{\varepsilon-1} \right\} \\
F_t &= C_t^{-\sigma} Y_t + \theta\beta \mathbb{E}_t[\Pi_{t+1}^{\varepsilon-1} F_{t+1}].
\end{aligned}$$

For S_t

$$\begin{aligned}
S_t &= C_t^{-\sigma} Y_t MC_t^r + \theta\beta \sum_{k=0}^{\infty}(\theta\beta)^k \mathbb{E}_t \left\{ C_{t+k+1}^{-\sigma} Y_{t+k+1} MC_{t+k+1}^r \left(\frac{P_{t+k+1}}{P_t}\right)^{\varepsilon} \left(\frac{P_{t+1}}{P_{t+1}}\right)^{\varepsilon} \right\} \\
S_t &= C_t^{-\sigma} Y_t MC_t^r + \theta\beta \mathbb{E}_t[\Pi_{t+1}^{\varepsilon} S_{t+1}].
\end{aligned}$$

2.2 Log-linearization

Taking a log-linear approximation at both side of (3)

$$\begin{aligned}
&\sum_{k=0}^{\infty}(\theta\beta)^k P^* P^{-1} C^{-\sigma} Y \mathbb{E}_t \left[1 + p_t^* - p_t - \sigma c_{t+k} + y_{t+k} + (\varepsilon-1)(p_{t+k} - p_t)\right] = \\
&\left(\frac{\varepsilon}{\varepsilon-1}\right) \sum_{k=0}^{\infty}(\theta\beta)^k C^{-\sigma} Y MC^r \mathbb{E}_t \left[1 - \sigma c_{t+k} + y_{t+k} + mc_{t+k}^r + (\varepsilon-1)(p_{t+k} - p_t)\right] \\
&\Rightarrow p_t^* = (1-\theta\beta) \sum_{k=0}^{\infty}(\theta\beta)^k \mathbb{E}_t[mc_{t+k}^r + p_{t+k}] \\
&p_t^* - p_{t-1} = (1-\theta\beta) \sum_{k=0}^{\infty}(\theta\beta)^k \mathbb{E}_t[mc_{t+k}^r + p_{t+k} - p_{t-1}].
\end{aligned}$$

Working the expression associated to prices

$$
\begin{aligned}
&(1-\theta\beta)\left[(\theta\beta)^0(p_t-p_{t-1})+(\theta\beta)^1(p_{t+1}-p_{t-1})+(\theta\beta)^2(p_{t+2}-p_{t-1})+\cdots\right]\\
&=(1-\theta\beta)\left[(\theta\beta)^0\pi_t+(\theta\beta)^1(\pi_t+\pi_{t+1})+(\theta\beta)^2(\pi_t+\pi_{t+1}+\pi_{t+2})+\cdots\right]\\
&=\left[(\theta\beta)^0\pi_t+(\theta\beta)^1(\pi_t+\pi_{t+1})+(\theta\beta)^2(\pi_t+\pi_{t+1}+\pi_{t+2})+\cdots\right]\\
&-\left[(\theta\beta)^1\pi_t+(\theta\beta)^2(\pi_t+\pi_{t+1})+(\theta\beta)^3(\pi_t+\pi_{t+1}+\pi_{t+2})+\cdots\right]\\
&=(\theta\beta)^0\pi_t+(\theta\beta)^1\pi_{t+1}+(\theta\beta)^2\pi_{t+2}+\cdots.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
p_t^*-p_{t-1}&=(1-\theta\beta)\sum_{k=0}^{\infty}(\theta\beta)^k\mathbb{E}_t[mc_{t+k}^r]+\sum_{k=0}^{\infty}(\theta\beta)^k\pi_{t+k}\\
&=(1-\theta\beta)mc_t^r+\pi_t+\theta\beta\mathbb{E}_t[p_t^*-p_{t-1}]\\
\pi_t&=\frac{(1-\theta\beta)(1-\theta)}{\theta}mc_t^r+\beta\mathbb{E}_t[\pi_{t+1}],
\end{aligned}
$$

where we use the fact that $\pi_t=(1-\theta)(p_t^*-p_{t-1})$. Note that this is the standard form of the NK curve in a one sector model.

3 Multisector economy with representative agent

3.1 Recursive formulation

In this case, the stochastic discount factor is the same as before. However, now prices are indexed by the sector of the economy, j, in which each firm belongs (the same holds for the updating probability, the sectoral price, the quantity produced and the marginal cost). In principle, this could also affect the elasticity of substitution but for now we keep it constant. This implies that equation (2) reads as

$$
\sum_{k=0}^{\infty}\theta_j^k\mathbb{E}_t\left\{Q_{t,t+k}\left[P_{jt}^*(z)P_{jt+k}^{\varepsilon}Y_{jt+k}-\left(\frac{\varepsilon}{\varepsilon-1}\right)MC_{jt+k}P_{jt+k}^{\varepsilon}Y_{jt+k}\right]\right\}=0. \tag{4}
$$

Following similar steps as before, (4) can be written as

$$
\sum_{k=0}^{\infty}(\theta_j\beta)^k\mathbb{E}_t\left\{C_{t+k}^{-\sigma}\frac{1}{P_{t+k}}\left[P_{jt}^*(z)P_{jt+k}^{\varepsilon}Y_{jt+k}-\left(\frac{\varepsilon}{\varepsilon-1}\right)MC_{jt+k}P_{jt+k}^{\varepsilon}Y_{jt+k}\right]\right\}=0.
$$

Note that the previous expression combines sectoral prices with the aggregate price coming from the consumption CPI into the stochastic discount factor. Dividing by P_t and P_{jt}^{ε}

$$
\frac{P_{jt}^*(z)}{P_t}\underbrace{\sum_{k=0}^{\infty}(\theta_j\beta)^k\mathbb{E}_t\left\{C_{t+k}^{-\sigma}Y_{jt+k}\left(\frac{P_{jt+k}}{P_{jt}}\right)^{\varepsilon}\frac{P_t}{P_{t+k}}\right\}}_{F_{jt}}=\left(\frac{\varepsilon}{\varepsilon-1}\right)\underbrace{\sum_{k=0}^{\infty}(\theta_j\beta)^k\mathbb{E}_t\left\{C_{t+k}^{-\sigma}Y_{jt+k}MC_{jt+k}^r\left(\frac{P_{jt+k}}{P_{jt}}\right)^{\varepsilon}\right\}}_{S_{jt}}. \tag{5}
$$

Again working each expression

$$
\begin{aligned}
F_{jt} &= C_t^{-\sigma} Y_t + \theta_j \beta \sum_{k=0}^{\infty} (\theta_j \beta)^k \mathbb{E}_t \left\{ C_{t+k+1}^{-\sigma} Y_{jt+k+1} \left(\frac{P_{jt+k+1}}{P_{jt}} \right)^{\varepsilon} \frac{P_t}{P_{t+k+1}} \left(\frac{P_{jt+1}}{P_{jt+1}} \right)^{\varepsilon} \left(\frac{P_{t+1}}{P_{t+1}} \right) \right\} \\
F_{jt} &= C_t^{-\sigma} Y_t + \theta_j \beta \mathbb{E}_t [\Pi_{jt+1}^{\varepsilon} \Pi_{t+1}^{-1} F_{jt+1}],
\end{aligned}
$$

where Π_{jt} denotes the sectoral (gross) inflation and Π_t is the aggregate inflation. For S_t

$$
\begin{aligned}
S_{jt} &= C_t^{-\sigma} Y_t MC_{jt}^r + \theta_j \beta \sum_{k=0}^{\infty} (\theta_j \beta)^k \mathbb{E}_t \left\{ C_{t+k+1}^{-\sigma} Y_{jt+k+1} MC_{jt+k+1}^r \left(\frac{P_{jt+k+1}}{P_{jt}} \right)^{\varepsilon} \left(\frac{P_{jt+1}}{P_{jt+1}} \right)^{\varepsilon} \right\} \\
S_{jt} &= C_t^{-\sigma} Y_t MC_{jt}^r + \theta_j \beta \mathbb{E}_t [\Pi_{jt+1}^{\varepsilon} S_{jt+1}]
\end{aligned}
$$

3.2 Log-linearization

Taking a log-linear approximation at every term in (5) we get

$$
\begin{aligned}
&\sum_{k=0}^{\infty} (\theta_j \beta)^k P_j^* P^{-1} C^{-\sigma} Y_j \mathbb{E}_t [1 + p_{jt}^* - p_t - \sigma c_{t+k} + y_{t+k} + \varepsilon (p_{jt+k} - p_{jt}) - (p_{t+k} - p_t)] = \\
&\left(\frac{\varepsilon}{\varepsilon - 1} \right) \sum_{k=0}^{\infty} (\theta_j \beta)^k C^{-\sigma} Y_j MC_j^r \mathbb{E}_t [1 - \sigma c_{t+k} + y_{t+k} + mc_{jt+k}^r + \varepsilon (p_{jt+k} - p_{jt})] \\
&\Rightarrow p_{jt}^* = (1 - \theta_j \beta) \sum_{k=0}^{\infty} (\theta_j \beta)^k \mathbb{E}_t [mc_{jt+k}^r + p_{t+k}] \\
&p_{jt}^* - p_{jt-1} = (1 - \theta_j \beta) \sum_{k=0}^{\infty} (\theta_j \beta)^k \mathbb{E}_t [mc_{jt+k}] + (1 - \theta_j \beta) \sum_{k=0}^{\infty} (\theta_j \beta)^k \mathbb{E}_t [p_{t+k} - p_{jt-1}]
\end{aligned}
$$

The latter expression can be written as (ignoring the expectation term for simplicity and recalling $p_{jt}^r = p_{jt} - p_t$)

$$
\begin{aligned}
&(1 - \theta_j \beta) \left[(\theta_j \beta)^0 (p_t - p_{jt-1}) + (\theta_j \beta)^1 (p_{t+1} - p_{jt-1}) + (\theta_j \beta)^2 (p_{t+2} - p_{jt-1}) + \cdots \right] \\
&= (1 - \theta_j \beta) \left[(\theta_j \beta)^0 (-p_{jt}^r + \pi_{jt}) + (\theta_j \beta)^1 (-p_{jt+1}^r + \pi_{jt} + \pi_{jt+1}) + \cdots \right] \\
&= \left[(\theta_j \beta)^0 \pi_{jt} + (\theta_j \beta)^1 \pi_{jt+1} + \cdots \right] - (1 - \theta_j \beta) \left[(\theta_j \beta)^0 p_{jt}^r + (\theta_j \beta)^1 p_{jt+1}^r + \cdots \right] \\
&= \sum_{k=0}^{\infty} (\theta_j \beta)^k \mathbb{E}_t [\pi_{jt+k}] - (1 - \theta_j \beta) \sum_{k=0}^{\infty} (\theta_j \beta)^k \mathbb{E}_t [p_{jt+k}^r].
\end{aligned}
$$

Then we have

$$
\begin{aligned}
p_{jt}^* - p_{jt-1} &= (1 - \theta_j \beta) \sum_{k=0}^{\infty} (\theta_j \beta)^k \mathbb{E}_t [mc_{jt+k} - p_{jt+k}^r] + \sum_{k=0}^{\infty} (\theta_j \beta)^k \mathbb{E}_t [\pi_{jt+k}] \\
&= (1 - \theta_j \beta)(mc_{jt}^r - p_{jt}^r) + \pi_{jt} + \theta_j \beta \left[(1 - \theta_j \beta) \sum_{k=0}^{\infty} (\theta_j \beta)^k \mathbb{E}_t [mc_{jt+k+1} - p_{jt+k+1}^r] + \sum_{k=0}^{\infty} (\theta_j \beta)^k \mathbb{E}_t [\pi_{jt+k+1}] \right] \\
&= (1 - \theta_j \beta)(mc_{jt}^r - p_{jt}^r) + \pi_{jt} + \theta_j \beta \mathbb{E}_t [p_{jt+1}^* - p_{jt}].
\end{aligned}
$$

Recall that $\pi_{jt} = (1-\theta_j)(p^*_{jt} - p_{jt-1})$ so

$$\pi_{jt} = \frac{(1-\theta_j\beta)(1-\theta_j)}{\theta_j}(mc^r_{jt} - p^r_{jt}) + \beta\mathbb{E}_t[\pi_{jt+1}]$$

4 Multisector economy with heterogeneous agents

4.1 Recursive formulation

In this case, there is no longer a single stochastic discount factor in the economy, so it is not obvious how firms discount the future. In this context, a natural simplifying assumption would be just to consider a common discount factor β. In this case, equation (2) would be

$$\sum_{k=0}^{\infty}(\theta_j\beta)^k\mathbb{E}_t\left\{P^*_{jt}(z)P^{\varepsilon}_{jt+k}Y_{jt+k} - \left(\frac{\varepsilon}{\varepsilon-1}\right)MC_{jt+k}P^{\varepsilon}_{jt+k}Y_{jt+k}\right\} = 0. \tag{6}$$

Dividing the whole expression by P_t and P^{ε}_{jt}, and dividing/multiplying the right-hand side by P_{t+k} we get

$$\frac{P^*_{jt}(z)}{P_t}\underbrace{\sum_{k=0}^{\infty}(\theta_j\beta)^k\mathbb{E}_t\left\{\left(\frac{P_{jt+k}}{P_{jt}}\right)^{\varepsilon}Y_{jt+k}\right\}}_{F_{jt}} = \left(\frac{\varepsilon}{\varepsilon-1}\right)\underbrace{\sum_{k=0}^{\infty}(\theta\beta)^k\mathbb{E}_t\left\{\left(\frac{P_{jt+k}}{P_{jt}}\right)^{\varepsilon}Y_{jt+k}MC^r_{jt+k}\frac{P_{t+k}}{P_t}\right\}}_{S_{jt}}. \tag{7}$$

Working each expression we get

$$\begin{aligned}
F_{jt} &= Y_{jt} + \theta_j\beta\sum_{k=0}^{\infty}(\theta_j\beta)^k\mathbb{E}_t\left\{\left(\frac{P_{jt+k+1}}{P_{jt}}\right)^{\varepsilon}Y_{jt+k+1}\left(\frac{P_{jt+1}}{P_{jt+1}}\right)^{\varepsilon}\right\}\\
F_{jt} &= Y_{jt} + \theta_j\beta\mathbb{E}_t[\Pi^{\varepsilon}_{jt+1}F_{jt+1}],
\end{aligned}$$

and

$$\begin{aligned}
S_{jt} &= Y_{jt}MC^r_{jt} + \theta_j\beta\sum_{k=0}^{\infty}(\theta\beta)^k\mathbb{E}_t\left\{\left(\frac{P_{jt+k+1}}{P_{jt}}\right)^{\varepsilon}Y_{jt+k+1}MC^r_{jt+k+1}\frac{P_{t+k+1}}{P_t}\left(\frac{P_{jt+1}}{P_{jt+1}}\right)^{\varepsilon}\frac{P_{t+1}}{P_{t+1}}\right\}\\
S_{jt} &= Y_{jt}MC^r_{jt} + \theta_j\beta\mathbb{E}_t[\Pi^{\varepsilon}_{jt+1}\Pi_{t+1}S_{jt+1}].
\end{aligned}$$

4.2 Log-linearization

Taking a log-linear approximation of (6)

$$\begin{aligned}
&\sum_{k=0}^{\infty}(\theta_j\beta)^kP^*_jP^{-1}Y_j\mathbb{E}_t[1 + p^*_{jt} - p_t + \varepsilon(p_{jt+k} - p_{jt}) + y_{jt+k}]\\
&= \left(\frac{\varepsilon}{\varepsilon-1}\right)\sum_{k=0}^{\infty}(\theta_j\beta)^kY_jMC^r_j\mathbb{E}_t[1 + \varepsilon(p_{jt+k} - p_{jt}) + y_{jt+k} + mc^r_{jt+k} + (p_{t+k} - p_t)]\\
&\Rightarrow p^*_{jt} = (1-\theta_j\beta)\sum_{k=0}^{\infty}(\theta_j\beta)^k\mathbb{E}_t[mc^r_{jt+k} + p_{t+k}].
\end{aligned}$$

Note that the latter expression equals the one for multisector economies with a representative household. Hence, the log-linear approximation is the same.

5 Price and price dispersion evolution

From the CES aggregation, recall that $P_t^{1-\varepsilon} = \theta P_{t-1}^{1-\varepsilon} + (1-\theta)(P_t^*)^{1-\varepsilon}$. Therefore, the price evolves as

$$1 = \theta \Pi_t^{\varepsilon-1} + (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{1-\varepsilon}. \tag{8}$$

On the other hand, from the definition of price dispersion

$$\begin{aligned}
\Delta_t = \int_0^1 \left(\frac{P_t(z)}{P_t}\right)^{-\varepsilon} dz &= \int_0^\theta \left(\frac{P_{t-1}(z)}{P_t}\right)^{-\varepsilon} dz + \int_\theta^1 \left(\frac{P_t^*}{P_t}\right)^{-\varepsilon} dz \\
&= \int_0^\theta \left(\frac{P_{t-1}(z)}{P_t}\right)^{-\varepsilon} dz + (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\varepsilon} \\
&= \int_0^\theta \left(\frac{P_{t-1}(z)}{P_t}\right)^{-\varepsilon} \left(\frac{P_{t-1}}{P_{t-1}}\right)^{-\varepsilon} dz + (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\varepsilon} \\
&= \theta \Pi_t^{\varepsilon} \int_0^1 \left(\frac{P_{t-1}(z)}{P_{t-1}}\right)^{-\varepsilon} dz + (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\varepsilon} \\
&= \theta \Pi_t^{\varepsilon} \Delta_{t-1} + (1-\theta)\left(\frac{P_t^*}{P_t}\right)^{-\varepsilon}.
\end{aligned}$$