Ito's Lemma

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This short note summarizes the sketch of derivation of Ito's lemma presented in Chapter 1 of Dixit (1993)

1 Random walk representation

Consider time periods of length Δt and steps of length Δh. The random walk variable Δx can take values $\pm\Delta h$ with probability p and $q = 1 - p$ (up and down, respectively). Then, the first moments are:

$$\begin{aligned}
\mathbb{E}(\Delta x) &= p(\Delta h) + q(-\Delta h) = (p-q)\Delta h \\
\mathbb{E}[(\Delta x)^2] &= p(\Delta h)^2 + q(-\Delta h)^2 = (p+q)(\Delta h)^2 = (\Delta h)^2 \\
\mathbb{V}(\Delta x) &= \mathbb{E}[(\Delta x)^2] - [\mathbb{E}(\Delta x)]^2 = (\Delta h)^2 - (p-q)^2(\Delta h)^2 = 4pq(\Delta h)^2
\end{aligned}$$

The time interval of length t has $n = t/\Delta t$ discrete steps. Since each step is independent of the previous one, the cumulative change $x_t - x_0$ has a binominal distribution with mean $n(p-q)\Delta h = t(p-q)/\Delta t$ and variance $4npq(\Delta h)^2 = 4tpq(\Delta h)^2/\Delta t$. This is, the same expectation and variance of the random walk but multiplied by the number of steps between the origina and the current realization.
Now consider $\Delta h = \sigma\sqrt{t}$, $p = \frac{1}{2}[1 + \frac{\mu}{\sigma}\sqrt{\Delta t}] = \frac{1}{2}[1 + \frac{\mu}{\sigma^2}\Delta h]$ and $q = \frac{1}{2}[1 - \frac{\mu}{\sigma}\sqrt{\Delta t}] = \frac{1}{2}[1 - \frac{\mu}{\sigma^2}\Delta h]$, where μ and σ are the mean and standard deviation of x, which is normally distributed. With this parametrization we have the following mean and variance for the cumulative change:

$$\begin{aligned}
&\text{Mean}: t(p-q)\frac{\Delta h}{\Delta t} = t(2p-1)\frac{\Delta h}{\Delta t} = t\left[2\frac{1}{2}\left[1 + \frac{\mu}{\sigma^2}\Delta h\right] - 1\right]\frac{\Delta h}{\Delta t} = t\frac{\mu}{\sigma^2}\sigma^2\frac{\Delta t}{\Delta t} = \mu t \\
&4pq = \left[1 + \frac{\mu}{\sigma^2}\Delta h\right]\left[1 - \frac{\mu}{\sigma^2}\Delta h\right] = 1 - \left(\frac{\mu}{\sigma}\right)^2\Delta t \\
&\text{Var}: 4tpq\frac{(\Delta h)^2}{\Delta t} = t\left[1 - \left(\frac{\mu}{\sigma}\right)^2\Delta t\right]\frac{(\Delta h)^2}{\Delta t} = t\left[1 - \left(\frac{\mu}{\sigma}\right)^2\Delta t\right]\sigma^2 \to \sigma^2 t
\end{aligned}$$

Therefore, the limit of the random walk, when both time length and step goes to zero, is a Brownian motion. The key element is to preserve the relation $\Delta h = \sigma\sqrt{\Delta t}$.

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2 Ito's lemma

Assume x_t follows a Brownian motion with parameters (μ, σ) and the following relation $y_t = f(x_t)$, where f is a given non-random twice.continuous differentiable function. Consider a Taylor expansion at the point $y_0 = f(x_0)$:

$$y_t - y_0 = f'(x_0)(x_t - x_0) + \frac{1}{2}f''(x_0)(x_t - x_0)^2 + \dots$$

Now, consider the expectation of this expression:

$$\begin{aligned}
\mathbb{E}(y_t - y_0) &= f'(x_0)\mathbb{E}(x_t - x_0) + \frac{1}{2}f''(x_0)\mathbb{E}[(x_t - x_0)^2] + \dots \\
&= f'(x_0)\mu t + \frac{1}{2}f''(x_0)[\sigma^2 t + \mu^2 t^2] + \dots \\
&= [f'(x_0)\mu + \frac{1}{2}f''(x_0)\sigma^2]t
\end{aligned}$$

where in the second line we use $\mathbb{E}[(x_t - x_0)^2] = \mathbb{V}(x_t - x_0) + [\mathbb{E}(x_t - x_0)]^2 = \sigma^2 t + \mu^2 t^2$ and in the third line the fact that t is small, so only first order terms are relevant.
The variance is:

$$\begin{aligned}
\mathbb{V}(y_t - y_0) &= \mathbb{V}\{f'(x_0)(x_t - x_0) + \frac{1}{2}f''(x_0)(x_t - x_0)^2 + \dots\} \\
&= [f'(x_0)]^2\mathbb{V}(x_t - x_0) = [f'(x_0)]^2\sigma^2 t
\end{aligned}$$

We want to apply the same idea to any infinitesimal increment dy in $y = f(x)$ in the interval dt. Therefore:

$$\begin{aligned}
\mathbb{E}(dy) &= [f'(x)\mu + \frac{1}{2}f''(x)\sigma^2]dt \\
\mathbb{V}(dy) &= [f'(x)]^2\sigma^2 dt
\end{aligned}$$

which is the same as before but for any x_t and using the right time interval dt.
Finally we say that the process of y is a *general diffusion process* of the form:

$$dy = [f'(x)\mu + \frac{1}{2}f''(x)\sigma^2]dt + f'(x)\sigma d\omega \tag{1}$$

The, equation (1) corresponds to the Ito's lemma.

References

Dixit, A. (1993). *The Art of Smooth Pasting*, Princeton University Press.