Exact hat algebra

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Summary This note summarizes the exact hat algebra technique using the Armington model of trade.

Baseline equations and definitions Consider a world with N countries, which are indexed by i and j. L_i is the labor endowment, w_i the endogenous labor wage. Income is given by $Y_i = w_i L_i$. Trade flows are given by X_{ij}. Parameters χ_i, τ_{ij} and ϵ are a productivity shifter, trade costs and elasticity of substitution, respectively. Finally define sales shares $\gamma_{ij} = X_{ij}/Y_i$.
The two relevant equations of this model are the market clearing condition and the gravity equations

$$w_i L_i = \sum_{j=1}^{N} \lambda_{ij} w_j L_j \tag{1}$$

$$\lambda_{ij} = \frac{\chi_i (\tau_{ij} w_i)^{-\epsilon}}{\sum_{l=1}^{N} \chi_l (\tau_{lj} w_l)^{-\epsilon}}. \tag{2}$$

Suppose a variable x in the initial equilibrium. After a change in fundamentals, such variable is now x'. Define the hat-form version of variable x as $\widehat{x} \equiv x'/x$. This is, the ratio between the changed variable and the original one. The goal is to write the original system of equations in hat-form, which can be done as follows.

1. Market clearing

   After shock this equations reads as

$$w_i' L_i' = \sum_{j=1}^{N} \lambda_{ij}' w_j' L_j'. \tag{1'}$$

   Dividing (1') with (1)

$$\frac{w_i' L_i'}{w_i L_i} = \widehat{w}_i \widehat{L}_i = \frac{\sum_{j=1}^{N} \lambda_{ij}' w_j' L_j'}{\sum_{j=1}^{N} \lambda_{ij} w_j L_j} = \frac{\sum_{j=1}^{N} X_{ij}'}{\sum_{j=1}^{N} \lambda_{ij} w_j L_j}$$
$$= \frac{\sum_{j=1}^{N} X_{ij}'}{w_i L_i} = \sum_{j=1}^{N} \frac{X_{ij}}{w_i L_i} \frac{X_{ij}'}{X_{ij}} = \sum_{j=1}^{N} \gamma_{ij} \widehat{X}_{ij}$$

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2. Gravity equation

After a shock, this equation reads (in general form) as

$$\lambda'_{ij} = \frac{\chi'_i(\tau'_{ij}w'_i)^{-\epsilon}}{\sum_{l=1}^{N}\chi'_l(\tau'_{lj}w'_l)^{-\epsilon}}. \tag{2'}$$

Dividing (2′) with (2)

$$\begin{aligned}
\frac{\lambda'_{ij}}{\lambda_{ij}} = \widehat{\lambda_{ij}} &= \widehat{\chi}_i(\widehat{\tau}_{ij}\widehat{w}_i)^{-\epsilon}\left(\frac{\sum_{l=1}^{N}\chi_l(\tau_{lj}w_l)^{-\epsilon}}{\sum_{l=1}^{N}\chi'_l(\tau'_{lj}w'_l)^{-\epsilon}}\right) \\
&= \widehat{\chi}_i(\widehat{\tau}_{ij}\widehat{w}_i)^{-\epsilon}\left(\frac{\sum_{l=1}^{N}\chi_l(\tau_{lj}w_l)^{-\epsilon}}{\chi_l(\tau_{lj}w_l)^{-\epsilon}}\right)\left(\frac{\chi_l(\tau_{lj}w_l)^{-\epsilon}}{\sum_{l=1}^{N}\chi'_l(\tau'_{lj}w'_l)^{-\epsilon}}\right) \\
&= \widehat{\chi}_i(\widehat{\tau}_{ij}\widehat{w}_i)^{-\epsilon}\left(\frac{1}{\lambda_{lj}}\right)\left(\frac{1}{\sum_{l=1}^{N}\frac{\chi'_l(\tau'_{lj}w'_l)^{-\epsilon}}{\chi_l(\tau_{lj}w_l)^{-\epsilon}}}\right) \\
&= \frac{\widehat{\chi}_i(\widehat{\tau}_{ij}\widehat{w}_i)^{-\epsilon}}{\sum_{l=1}^{N}\lambda_{lj}\widehat{\chi}_l(\widehat{\tau}_{lj}\widehat{w}_l)^{-\epsilon}}
\end{aligned}$$

Summarizing, we have the following system of equations

$$\widehat{w}_i\widehat{L}_i = \sum_{j=1}^{N}\gamma_{ij}\widehat{X}_{ij} \tag{3}$$

$$\widehat{\lambda_{ij}} = \frac{\widehat{\chi}_i(\widehat{\tau}_{ij}\widehat{w}_i)^{-\epsilon}}{\sum_{l=1}^{N}\lambda_{lj}\widehat{\chi}_l(\widehat{\tau}_{lj}\widehat{w}_l)^{-\epsilon}}. \tag{4}$$

Suppose we are interested in the effect that a trade cost shock ($\widehat{\tau}_{ij}$) has over wages ($\widehat{w}_i$) and trade ($\widehat{\lambda}_{ij}$). By assumption, all other variables remain constant at the initial equilibrium, so from (3) we get $\widehat{w}_i = \sum_{j=1}^{N}\gamma_{ij}\widehat{X}_{ij} = \sum_{j=1}^{N}\gamma_{ij}\widehat{\lambda}_{ij}\widehat{w}_j$. Replacing the gravity equation we get

$$\widehat{w}_i = \sum_{j=1}^{N}\gamma_{ij}\widehat{w}_j\left(\frac{\widehat{\chi}_i(\widehat{\tau}_{ij}\widehat{w}_i)^{-\epsilon}}{\sum_{l=1}^{N}\lambda_{lj}\widehat{\chi}_l(\widehat{\tau}_{lj}\widehat{w}_l)^{-\epsilon}}\right) = \sum_{j=1}^{N}\frac{\gamma_{ij}\widehat{w}_j(\widehat{\tau}_{ij}\widehat{w}_i)^{-\epsilon}}{\sum_{l=1}^{N}\lambda_{lj}(\widehat{\tau}_{lj}\widehat{w}_l)^{-\epsilon}}.$$

Therefore, using data to pin-down ϵ and the initial equilibrium shares λ_{ij} and γ_{ij}, we can solve the previous $N \times N$ system of equations to get $\widehat{w}_i$. With this, we can recover $\widehat{\lambda}_{ij}$ from the gravity equation.