Implicitly Additive Non-homothetic CES

Damian Romero*

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This note summarizes the main properties and derivations of the Implicitly Additive Non-Homothetic CES demands by Hanoch (1975). These preferences have been used recently by Comin et al. (2019), Cravino and Sotelo (2019), Matsuyama (2019) and Redding and Weinstein (2020), among others, mostly in the context of trade and structural change. Most of the derivations come from Comin et al. (2015, 2018, 2019) and Redding and Weinstein (2020).[1]

1 Static problem

Consider the following generalized form for preferences that depend on the consumption of $i = 1, \ldots, I$ goods

$$1 = \sum_{i=1}^{I} (\omega_i C^{\epsilon_i})^{\frac{1}{\sigma}} C_i^{\frac{\sigma-1}{\sigma}}, \tag{1}$$

where ω_i is a taste parameter, $\sigma > 0$ is the constant elasticity of substitution between varieties, ϵ_i is the constant elasticity of consumption of variety i with respect to the consumption index C that allows preferences to be non-homothetic.

1.1 General results

1.1.1 Main derivations

While in the end we are going to use (1), for now consider a more general specification given by

$$1 = \sum_{i=1}^{I} f_i(U)^{\frac{1}{\sigma}} C_i^{\frac{\sigma-1}{\sigma}}. \tag{1'}$$

The utility maximization problem subject to the definition of the aggregator (1′) and total expenditure is

*Email: dromeroc@bcentral.cl

[1] See also Matsuyama's lecture "The Generalized Engel's Law: In Search for A New Framework" http://faculty.wcas.northwestern.edu/~kmatsu/Generalized%20Engel%27s%20Law.pdf.

$$\mathcal{L} = U + \rho\left(1 - \sum_{i=1}^{I} f_i(U)^{\frac{1}{\sigma}} C_i^{\frac{\sigma-1}{\sigma}}\right) + \lambda\left(E - \sum_{i=1}^{I} P_i C_i\right),$$

where ρ and λ are Lagrange multipliers and E corresponds to the total expenditures. The first order condition with respect to good i is

$$\lambda P_i = \rho\left(\frac{1-\sigma}{\sigma}\right) f_i(U)^{\frac{1}{\sigma}} C_i^{-\frac{1}{\sigma}}$$

which can be written as

$$P_i C_i = \frac{\rho}{\lambda}\left(\frac{1-\sigma}{\sigma}\right) f_i(U)^{\frac{1}{\sigma}} C_i^{\frac{\sigma-1}{\sigma}}.$$

Replacing in total expenditures

$$E = \sum_{i=1}^{I} \frac{\rho}{\lambda}\left(\frac{1-\sigma}{\sigma}\right) f_i(U)^{\frac{1}{\sigma}} C_i^{\frac{\sigma-1}{\sigma}} = \frac{\rho}{\lambda}\left(\frac{1-\sigma}{\sigma}\right) \sum_{i=1}^{I} f_i(U)^{\frac{1}{\sigma}} C_i^{\frac{\sigma-1}{\sigma}} = \frac{\rho}{\lambda}\left(\frac{1-\sigma}{\sigma}\right),$$

where the last equality uses the definition of the aggregator (1′).
Using total expenditures and the first order condition

$$s_i \equiv \frac{P_i C_i}{E} = f_i(U)^{\frac{1}{\sigma}} C_i^{\frac{\sigma-1}{\sigma}}.$$

Re-ordering this last expression we get the demand for good i and the expenditure share, which are given by

$$C_i = \left(\frac{E}{P_i}\right)^{\sigma} f_i(U) \tag{2}$$

$$s_i = \left(\frac{P_i}{E}\right)^{1-\sigma} f_i(U). \tag{3}$$

Using (2), we get total expenditures as

$$E = \sum_{i=1}^{I} P_i C_i = \sum_{i=1}^{I} P_i \left(\frac{E}{P_i}\right)^{\sigma} f_i(U) \quad \rightarrow \quad E^{1-\sigma} = \sum_{i=1}^{I} P_i^{1-\sigma} f_i(U). \tag{4}$$

1.1.2 Elasticities

Consider first the elasticity of expenditures with respect to utility, $\eta_U^E \equiv \frac{\partial E}{\partial U}\frac{U}{E}$. Using (4), the partial derivative of expenditures with respect to utility is

$$\frac{\partial E}{\partial U} = \frac{\partial \left[\sum_{i=1}^{I} P_i^{1-\sigma} f_i(U)\right]^{\frac{1}{1-\sigma}}}{\partial U} = \frac{1}{1-\sigma} E^{\sigma} \left(\sum_{i=1}^{I} P_i^{1-\sigma} \frac{\partial f_i(U)}{\partial U}\right).$$

Multiplying the previous expression by U/E we get

$$\eta_U^E = \frac{1}{1-\sigma} E^{\sigma-1} U \left(\sum_{i=1}^{I} P_i^{1-\sigma} \frac{\partial f_i(U)}{\partial U} \right) = \frac{1}{1-\sigma} \sum_{i=1}^{I} \underbrace{\left(\frac{P_i}{E} \right)^{1-\sigma} f_i(U)}_{s_i \text{ from (3)}} \underbrace{\frac{\partial f_i(U)}{\partial U} \frac{U}{f_i(U)}}_{=\eta_U^{f_i}}$$

$$\eta_U^E = \frac{1}{1-\sigma} \sum_{i=1}^{I} s_i \eta_U^{f_i} = \frac{\overline{\eta}_U^f}{1-\sigma}. \tag{5}$$

Using (2) we have the following relative demands

$$\frac{C_i}{C_j} = \left(\frac{P_j}{P_i} \right)^{\sigma} \frac{f_i(U)}{f_j(U)}.$$

Therefore, the elasticity of substitution ($\eta_{P_j/P_i}^{C_i/C_j}$) and the elasticity of relative demand with respect to utility ($\eta_U^{C_i/C_j}$) are

$$\eta_{P_j/P_i}^{C_i/C_j} \equiv \frac{\partial \log(C_i/C_j)}{\partial P_j/P_i} = \sigma \tag{6}$$

$$\eta_U^{C_i/C_j} \equiv \frac{\partial \log(C_i/C_j)}{\partial U} = \frac{\partial \log(f_i/f_j)}{\partial U} \tag{7}$$

$$\eta_E^{C_i} \equiv \frac{\partial \log C_i}{\partial \log E} = \sigma + \frac{\partial \log f_i(U)}{\partial \log U} \frac{\partial \log U}{\partial \log E} = \sigma + (1-\sigma) \frac{\eta_U^{f_i}}{\overline{\eta}_U^f}, \tag{8}$$

where the last equality in (8) is given by (the inverse of) (5).

1.2 Income isoelastic case

Working directly with aggregate consumption instead of utility and defining $f_i(U) = f_i(C) = \omega_i C^{\epsilon_i}$, which corresponds to (1). Note that in this case, by choosing $\epsilon_i = 1 - \sigma$ for every i, we recover the standard homothetic CES preferences.
Using this specification and all the previous results we get the demand for good i and its relative expenditure share

$$C_i = \omega_i \left(\frac{E}{P_i} \right)^{\sigma} C^{\epsilon_i} = \omega_i \left(\frac{P_i}{P} \right)^{-\sigma} C^{\epsilon_i + \sigma} \tag{9}$$

$$s_i = \omega_i \left(\frac{P_i}{E} \right)^{1-\sigma} C^{\epsilon_i} = \omega_i \left(\frac{P_i}{P} \right)^{1-\sigma} C^{\epsilon_i - (1-\sigma)}. \tag{10}$$

The expenditure function and the aggregate price are

$$E = \left(\sum_{i=1}^{I} \omega_i P_i^{1-\sigma} C^{\epsilon_i} \right)^{\frac{1}{1-\sigma}}$$

$$P = \frac{1}{C} \left(\sum_{i=1}^{I} \omega_i P_i^{1-\sigma} C^{\epsilon_i} \right)^{\frac{1}{1-\sigma}}.$$

In terms of elasticities we have

$$
\begin{aligned}
\eta_U^{f_i} &= \eta_C^{f_i} = \epsilon_i \\
\eta_U^E &= \eta_C^E = \frac{1}{1-\sigma}\sum_{i=1}^{I} s_i \eta_U^{f_i} = \frac{1}{1-\sigma}\sum_{i=1}^{I} s_i \epsilon_i = \frac{\overline{\epsilon}}{1-\sigma} \\
\eta_E^{C_i} &= \sigma + (1-\sigma)\frac{\epsilon_i}{\overline{\epsilon}}.
\end{aligned}
$$

As Engel aggregation requires, the income elasticities average to 1 when sectoral weights are given by expenditure shares, $\sum_i s_i \eta_E^{C_i}$. If good i has an income elasticity parameter ϵ_i that exceeds (is less than) the consumer's average elasticity parameter $\overline{\epsilon}$, then good i is a luxury (necessity) good, in the sense that it has an expenditure elasticity greater (smaller) than 1 at that point in time. This implies that being a luxury or necessity good is not an intrinsic characteristic of a good, but rather depends on the consumer's current composition of consumption expenditures and, ultimately, income.

2 Dynamic problem

The problem of the representative consumer is to maximize utility over time by choosing consumption and assets. Formally, the optimization problem is

$$
\max_{\{C_t, A_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\varsigma} - 1}{1-\varsigma} \right) \quad \text{subject to} \quad E(C_t) + A_{t+1} = w_t + R_{t-1}A_t,
$$

where $E(C_t)$ denotes expenditures given a level of consumption C_t. Letting λ_t be Lagrange multiplier of the problem, the first order conditions with respect to consumption and assets are

$$
\begin{aligned}
C_t^{-\varsigma} &= \lambda_t \frac{\partial E(C_t)}{\partial C_t} \\
\lambda_t &= \beta R_t \lambda_{t+1}.
\end{aligned}
$$

From previous results, we know that $\eta_C^E = \frac{\partial E}{\partial C}\frac{C}{E} = \frac{\overline{\epsilon}}{1-\sigma}$, so $\frac{\partial E}{\partial C} = \frac{\overline{\epsilon}}{1-\sigma}\frac{E}{C} = \frac{\overline{\epsilon}}{1-\sigma}P$. Combining these results we get the following Euler equation

$$
1 = \beta R_t \frac{\lambda_{t+1}}{\lambda_t} = \beta R_t \frac{C_{t+1}^{-\varsigma}\frac{1-\sigma}{\overline{\epsilon}_{t+1}}\frac{1}{P_{t+1}}}{C_t^{-\varsigma}\frac{1-\sigma}{\overline{\epsilon}_t}\frac{1}{P_t}} = \beta R_t \left(\frac{C_{t+1}}{C_t}\right)^{-\varsigma} \frac{\overline{\epsilon}_t}{\overline{\epsilon}_{t+1}}\frac{P_t}{P_{t+1}},
$$

which can also be written as

$$
1 = \beta R_t \left(\frac{C_{t+1}}{C_t}\right)^{1-\varsigma} \frac{\overline{\epsilon}_t}{\overline{\epsilon}_{t+1}}\frac{E_t}{E_{t+1}}.
$$

This is a nonlinear relationship between real and nominal consumption at all levels of income, and the household incorporate this in the intertemporal allocation problem. There is a wedge between marginal cost of real consumption and aggregate price index. The size of this wedge depends on the average income elasticity across sectors, which varies over time ($\overline{\epsilon}$).

3 Comparison with other preferences

One natural candidate for non-homothetic preferences are the Stone-Geary preferences, described by

$$U(C) = \sum_{i=1}^{I} \omega_i \frac{(C_i - \overline{C}_i)^{1-1/\sigma}}{1-1/\sigma}.$$

Some comments about them are in order

1. While the demand for each good depends on income (or total desired consumption), the marginal propensity to consume is independent of income. This allows for aggregation across households and we can talk about the representative household, even if they differ in their income and expenditure.

2. These preferences are asymptotically homothetic, suggesting that non-homotheticities are important only for poor households in poor countries.

3. The key parameters $\overline{C}_i$ are given by a quantity of good i, hence not unit-free. One can choose a unit of each good such that $\overline{C}_i = -1, 0, 1$ without loos of generality. In other words, it cannot meaningfully distinguish more than three sectors in terms of their income elasticities.

4. The marginal cost of consumption does not depend on income (the average cost does). However, the dependence of this price index on nominal income vanishes as the latter grows to infinity.

5. The elasticities of substitution between goods and income are (multiplicatively) related. Therefore, when income goes to infinity, the elasticity of substitution converges to a constant and the income elasticity converges to one.

References

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