A note on the properties of CES utility function: Elasticities

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December, 2016

In this note I summarize and derive some basic properties of elasticities for the CES utility function .

1 General form

Consider the following general form for the flow utility function of a representative consumer

$$U(C_t, L_t) = \frac{C_t^{1-\sigma} - 1}{1-\sigma} + \frac{L_t^{1+\eta}}{1+\eta} = u(C_t) + v(N_t) \tag{1}$$

Given this utility function, we will derive two main elasticities typically used in economics.

2 Intertemporal elasticity of substitution (IES)

Corresponds to a measure of the curvature of the utility function that captures the willingness to accept changes in consumption pattern over time. The larger the curvature, the more preferable is to smooth the consumption pattern.

The general form is

$$IES = -\frac{U'(C_t)}{U''(C_t)C_t}$$

and in particular for (1)

$$IES = -\frac{C_t^{-\sigma}}{-\sigma C_t^{-\sigma-1}} = \frac{1}{\sigma}$$

which gives the name of constant elasticity of substitution (in this case, between consumption in different periods).
Note also that when we consider uncertainty in the consumer problem, σ characterizes the degree of risk-aversion, which is constant across different levels of income. Given this, (1) is also known as constant-relative-risk-aversion (CRRA) utility function.[1]

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[1]Recall that the relative risk-aversion is computed as $-\frac{U''(C_t)C_t}{U'(C_t)}$, which in this case is just the inverse of the intertemporal elasticity of substitution.

3 Frisch elasticity

This measures the elasticity of labor supply with respect to wage, keeping marginal utility of wealth constant. In general is a measure of the substitution effect. To motivate it, consider a consumer that wants to maximize utility subject to an intertemporal budget constraint. Given this, the intratemporal equation is

$$\frac{v'(L_t)}{u'(C_t)} = w_t$$

where implicitly we have assumed that the numeraire good is the consumption good.
Leaving fixed the marginal utility of consumption, previous equation defines implicitly the dependance of labor supply with respect to wages. Differentiating the previous expression we have

$$\frac{v''(L_t)L_t'(w_t)}{u'(C_t)} = 1$$

We can work this expression as follows

$$\begin{aligned}
\frac{v'(L_t)}{L_t}\frac{v''(L_t)L_t'(w_t)}{u'(C_t)} &= \frac{v'(L_t)}{L_t} \\
\frac{v'(L_t)}{L_t}\frac{L_t'(w_t)}{u'(C_t)} &= \frac{v'(L_t)}{v''(L_t)L_t}
\end{aligned}$$

The Frisch elasticity (FE) of substitution is the right-hand side expression of previous equation. This is

$$FE = \frac{v'(L_t)}{v''(L_t)L_t}$$

For (1) we have

$$FE = \frac{v'(L_t)}{v''(L_t)L_t} = \frac{L_t^{\eta}}{L_t^{\eta-1}\eta} = \frac{1}{\eta}$$