A note on the properties of CES utility function: Consumption varieties

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November, 2016

In this note I summarize the properties of the constant elasticity of substitution (CES) utility function. This is typically used in the characterization of the New Keynesian model, where households consume a bundle of goods.

1 CES utility and variety demands

Consider the utility function

$$U = \left[\int_{\omega \in \Omega} q(\omega)^{\rho} d\omega \right]^{1/\rho}$$

where we consider a bundle of differentiated goods ω in the set Ω. We say also that the total level of consumption equals utility, so $Q = U$. The problem of the household is to minimize expenditure given a certain level of resources. This is, minimize the level of expenditure subject to $Z = \int_{\omega \in \Omega} p(\omega) q(\omega) d\omega$, where Z is the level of available income. Formally, the problem can be written as

$$\min_{q(\omega)} PQ \quad \text{subject to} \quad Z = \int_{\omega \in \Omega} p(\omega) q(\omega) d\omega$$

For any ω the first order condition is

$$P \left[\int_{\omega \in \Omega} q(\omega)^{\rho} d\omega \right]^{1/\rho - 1} q(\omega)^{\rho - 1} = p(\omega)$$

which can be re-arranged as follows

$$\begin{aligned} q(\omega)^{\rho-1} &= \frac{p(\omega)}{P} \left[\int_{\omega \in \Omega} q(\omega)^{\rho} d\omega \right]^{\frac{\rho-1}{\rho}} \\ q(\omega) &= \left[\frac{p(\omega)}{P} \right]^{\frac{1}{\rho}} \left[\int_{\omega \in \Omega} q(\omega)^{\rho} d\omega \right]^{\frac{1}{\rho}} \\ q(\omega) &= \left[\frac{p(\omega)}{P} \right]^{\frac{1}{\rho}} Q \end{aligned} \tag{1}$$

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The CES utility function is increasing in its arguments. Therefore, this preference is locally non-satiated and the household would like to spend all its income on consumption. Then we can replace in the restriction as follows

$$
\begin{aligned}
PQ &= \int_{\omega\in\Omega} p(\omega)q(\omega)d\omega = \int_{\omega\in\Omega} p(\omega)\left[\frac{p(\omega)}{P}\right]^{\frac{1}{\rho}} Qd\omega \\
P &= \left[\int_{\omega\in\Omega} p(\omega)^{\frac{\rho}{\rho-1}}\right]^{\frac{\rho-1}{\rho}}
\end{aligned}
$$

Therefore, we have defined the aggregate price level.

2 Income and total expenditure

Now we show that the household will spend total income. This is, $Z = PQ$. For arbitrary $\omega \in \Omega$ we have

$$
p(\omega)q(\omega) = p(\omega)\left[\frac{p(\omega)}{P}\right]^{\frac{1}{\rho}} Q = p(\omega)^{1-\sigma}P^{\sigma}Q
$$

where $\sigma \equiv \frac{1}{1-\rho}$. Integrating across all varieties:

$$
\begin{aligned}
Z &= \int_{\omega\in\Omega} p(\omega)q(\omega)d\omega = \int_{\omega\in\Omega} p(\omega)^{1-\sigma}P^{\sigma}Qd\omega = P^{\sigma}Q\left[\int_{\omega\in\Omega} p(\omega)^{1-\sigma}d\omega\right] \\
Z &= P^{\sigma}QP^{1-\sigma} = PQ
\end{aligned}
$$

3 Price-elasticity and elasticity of substitution

3.1 Price-elasticity

From demand 1 we can compute the price-elasticity

$$
\frac{\partial \log q(\omega)}{\partial \log p(\omega)} = \frac{\partial}{\partial \log p(\omega)}\left\{\left(\frac{1}{\rho-1}\right)[\log p(\omega) - \log P] + \log Q\right\} = \frac{1}{\rho-1}
$$

where we assume that the number of varieties is large enough, such that a change in the price of any ω does not change the aggregate level of prices.

3.2 Elasticity of substitution

Again from 1 we can compute the effect of a change in the price of variety ω' on the quantity demanded of variety ω. The relative demand is

$$
\frac{q(\omega)}{q(\omega')} = \left[\frac{p(\omega)}{p(\omega')}\right]^{\frac{1}{\rho-1}}
$$

Then, the elasticity of substitution is

$$\frac{\partial \log q(\omega)}{\partial \log p(\omega')} = \frac{\partial}{\partial \log p(\omega')} \left\{ \left(\frac{1}{\rho - 1} \right) [\log p(\omega) - \log p(\omega')] \right\} = \frac{1}{1-\rho}$$

and this is exactly the value $\sigma = \frac{1}{1-\rho}$. Note that this value is constant and that is the justification of the name of the utility function.

4 Welfare

As in the case with any preference, welfare can be measured with the indirect utility function. First note that the demand function can be written as

$$q(\omega) = \left[\frac{p(\omega)}{P} \right]^{-\sigma} Q = \left[\frac{p(\omega)}{P} \right]^{-\sigma} \frac{QP}{P} = \frac{p(\omega)^{-\sigma} Z}{P^{1-\sigma}}$$

For this particular case, we have that the indirect utility function (V) is

$$V = \left[\int_{\omega\in\Omega} \left\{ \frac{p(\omega)^{-\sigma} Z}{P^{1-\sigma}} \right\}^{\rho} d\omega \right]^{1/\rho} = \frac{Z}{P^{1-\sigma}} \left[\int_{\omega\in\Omega} p(\omega)^{-\sigma\rho} d\omega \right]^{1/\rho} = \frac{Z}{P^{1-\sigma}} \underbrace{\left[\int_{\omega\in\Omega} p(\omega)^{1-\sigma} d\omega \right]^{\frac{\sigma}{\sigma-1}}}_{P^{-\sigma}} = \frac{Z}{P}$$

Hence, welfare is always equal to expenditure over aggregate price.