Affine Yield Models–Pricing Recursion

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This note shows how to construct the recursion for no-arbitrage bond pricing. First I show a canonical model to then present how to construct the recursion to price all kinds of securities. These results can be easily extended to more complex settings.[1]

1 State variables dynamics and pricing kernel

Consider a Gaussian term structure model with $K \times 1$ pricing factors that evolve under the physical measure $\mathbb{P}$ according to the following autoregression

$$X_{t+1} = (I_K - \Phi)\mu_x + \Phi X_t + \nu_{t+1}, \tag{1}$$

where ν are iid Gaussian with $E_t[\nu_{t+1}] = 0_{K\times 1}$, $V_t[\nu_{t+1}] = \Sigma$, and where $E_t(\cdot)$ and $V_t(\cdot)$ denote the conditional expectation and the conditional variance, respectively.
Assume also that assets are priced by the stochastic discount factor

$$M_{t+1} = \exp\left(-r_t + \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\Sigma^{-1/2}\nu_{t+1}\right), \tag{2}$$

where r_t is the nominal short rate. In this specification, the $K \times 1$ price of risk vector λ_t takes the essentially affine form

$$\lambda_t = \Sigma^{-1/2}(\lambda_0 + \lambda_1 X_t). \tag{3}$$

For convenience, define

$$\widetilde{\mu} = (I_K - \Phi)\mu_x - \lambda_0 \tag{4}$$
$$\widetilde{\Phi} = \Phi - \lambda_1. \tag{5}$$

These parameters govern the dynamics of the pricing factors under the risk-neutral or pricing measure ($\mathbb{Q}$) and feature prominently in the recursive pricing relationships derived in the following section.

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[1]For further details, see also Backus et.al (1998)'s "Discrete-time Models for Bond Pricing" and Appendix C in Bolder (2001)'s "Affine Term-Structure Models: Theory and Implementation".

2 No-arbitrage pricing

In Gaussian affine term structure models the log price, P_t^n, of a risk-free discount bond with remaining time to maturity n, takes the form

$$\log P_t^n = p_t^n = A_n + B_n' X_t, \tag{6}$$

which implies that

$$r_t = \delta_0 + \delta_1' X_t. \tag{7}$$

By no-arbitrage, the following relation must hold for any n and any t

$$\begin{aligned} P_t^n &= E_t[M_{t+1} P_{t+1}^{n-1}] \\ \exp(p_t^n) &= E_t[\exp(m_{t+1} + p_{t+1}^{n-1})]. \end{aligned} \tag{8}$$

We must show that (6) holds for any maturity n. To do this, we follow a guess-and-verify approach. First, replace the guess in the right-hand side of (8)

$$\exp(p_t^n) = E_t[\exp(m_{t+1} + A_{n-1} + B_{n-1}' X_{t+1})].$$

We are going to assume that the stochastic discount factor and the pricing factors are jointly log-normal. In that case, the previous expression can be written as

$$\exp(p_t^n) = \exp\left[E_t\left(m_{t+1} + A_{n-1} + B_{n-1}' X_{t+1}\right) + \frac{1}{2} V_t\left(m_{t+1} + A_{n-1} + B_{n-1}' X_{t+1}\right)\right]. \tag{9}$$

From previous expression, we need to know the conditional expectation of the stochastic discount factor and the pricing factors. Also, from the properties of the variance, we need to know variance of the stochastic discount factor, the variance of the pricing factors and the covariance between all these terms. We are going to work each expression in turn

Stochastic discount factor Taking logs on each side of (2), we have

$$\begin{aligned} E_t[m_{t+1}] &= -r_t - \frac{1}{2}\lambda_t' \lambda_t \\ V_t[m_{t+1}] &= \lambda_t' \Sigma^{-1/2} \Sigma \Sigma^{-1/2} \lambda_t = \lambda_t' \lambda_t. \end{aligned}$$

Pricing factors Using the law of motion of factors (1) and pre-multiplying by B_{n-1}' we have

$$\begin{aligned} E_t[B_{n-1}' X_{t+1}] &= B_{n-1}'(I_K - \Phi)\mu_x + B_{n-1}' \Phi X_t \\ V_t[B_{n-1}' X_{t+1}] &= B_{n-1}' \Sigma B_{n-1}. \end{aligned}$$

Covariance Recall that the conditional covariance between a variable y and z can be computed as $cov_t[y,z] = E_t[(y - E_t(y))(z - E_t(z))]$. Using previous results, we have

$$\begin{aligned} cov_t[m_{t+1}, B'_{n-1}X_{t+1}] &= E_t[(-\lambda'_t\Sigma^{-1/2}\nu_{t+1})(B'_{n-1}\nu_{t+1})'] \\ &= -\lambda'_t\Sigma^{-1/2}\Sigma B_{n-1} \\ &= -(\lambda_0 + \lambda_1 X_t)'B_{n-1}. \end{aligned}$$

Combining all previous elements we have

$$E_t\left(m_{t+1} + A_{n-1} + B'_{n-1}X_{t+1}\right) = -r_t - \frac{1}{2}\lambda'_t\lambda_t + A_{n-1} + B'_{n-1}(I_K - \Phi)\mu_x + B'_{n-1}\Phi X_t \tag{10}$$

$$V_t\left(m_{t+1} + A_{n-1} + B'_{n-1}X_{t+1}\right) = \lambda'_t\lambda_t + B'_{n-1}\Sigma B_{n-1} - 2(\lambda_0 + \lambda_1 X_t)'B_{n-1}. \tag{11}$$

We can replace (10) and (11) in (9). Using (7) to get an expression for the short-term rate and re-arranging we have

$$\begin{aligned} \exp(p^n_t) = \exp\Big[\Big\{ &-\delta_0 + A_{n-1} + B'_{n-1}(I_K - \Phi)\mu_x + \frac{1}{2}B_n - 1'\Sigma B_{n-1} - B'_{n-1}\lambda_0\Big\} \\ &+ \Big\{ -\delta'_1 + B'_{n-1}\Phi - B'_{n-1}\lambda_1\Big\}X_{t+1}\Big]. \end{aligned}$$

Because the guess must hold for any maturity, matching coefficients and using the definition of $\widetilde{\mu}$ and $\widetilde{\Phi}$ we have

$$\begin{aligned} A_n &= A_{n-1} + B'_{n-1}\widetilde{\mu} + \frac{1}{2}B'_{n-1}\Sigma B_{n-1} - \delta_0 \\ B_n &= B'_{n-1}\widetilde{\Phi} - \delta'_1, \end{aligned}$$

which proofs the recursive structure of the model.