Derivation of Atkenson and Burstein (2008)

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1 A direct derivation from AB (2008)

Summary Consider a two-country model indexed by i. The representative household derives utility from an aggregate consumption (final) good c_i. this is composed by a continuum of sectoral goods indexed by j, which are combined in a CES function. At the same time, each sector is composed by $k = 1, \ldots, K$ firms that produce using only labor. In this note, I derive the markup equation for firms presented in the paper.

Final consumption This good is produced using output from sectors indexed by j

$$c_i = \left[\int_0^1 y_{ij}^{1-1/\eta} dj\right]^{\eta/(\eta-1)}.$$

From the expenditure minimization problem, we can get a demand for each sector and the ideal price index for the final good

$$\frac{P_{ij}}{P_i} = \left(\frac{y_{ij}}{c_i}\right)^{-1/\eta} \tag{1}$$

$$P_i = \left[\int_0^1 P_{ij}^{1-\eta} dj\right]^{1/(1-\eta)}$$

Sectoral good Each sectoral good is produced by combining the product of a finite number of firms

$$y_{ij} = \left[\sum_{k=1}^{K} q_{ijk}^{(\rho-1)/\rho}\right]^{\rho/(\rho-1)}, \tag{2}$$

where q_{ijk} denotes sales in country i of firm k in sector j. From this problem, get a demand for firms and the ideal price index for the sectoral good as

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$$\frac{P_{ijk}}{P_{ij}} = \left(\frac{q_{ijk}}{y_{ij}}\right)^{-1/\rho} \tag{3}$$
$$P_{ij} = \left[\sum_{k=1}^{K} P_{ijk}^{1-\rho}\right]^{1/(1-\rho)}$$

Firm's problem Each firm produces with a constant returns technology using only labor. Therefore, the marginal cost of a firm k in country i, sector j is $W_i/(A_i z_{jk})$, where W is the wage (common to all firms in the country), A is a country-aggregate productivity shock, and z is a firm-sector specific productivity shock. In what follows, this terms is just MC_{ijk}. A firm k must choose its price and quantity produced, taking as given productivity, wage, the rest of quantities produced, aggregate consumption/price. This is, it must solve

$$\max_{P_{ijk}, q_{ijk}} P_{ijk} q_{ijk} - q_{ijk} MC_{ijk},$$

subject to (1) and (3).
Note that the combined demand (equations (1) and (3)) can be written as $P_{ijk} = P_i(c_i^{1/\eta})(q_{ijk}^{-1/\rho})(y_{ij}^{1/\rho-1/\eta})$. For convenience, first note

$$\frac{\partial y_{ij}}{\partial q_{ijk}} = y_{ij}\left(\frac{q_{ijk}^{1-1/\rho}}{\sum_{l=1}^{K} q_{ijl}^{1-1/\rho}}\right) q_{ijk}^{-1}.$$

Replacing demand in the objective function, gets the following first order condition with respect to q_{ijk}

$$P_i(c_i^{1/\eta})\left[\left(\frac{\rho-1}{\rho}\right) q_{ijk}^{-1/\rho} y_{ij}^{1/\rho-1/\eta} + q_{ijk}^{-1/\rho}\left(\frac{1}{\rho}-\frac{1}{\eta}\right) y_{ij}^{1/\rho-1/\eta-1}\frac{\partial y_{ij}}{\partial q_{ijk}}\right] = MC_{ijk}.$$

By replacing the partial derivative of sectoral output with respect to firm k's output, and the demand for sectors, the previous equation can be written as

$$\left(\frac{\rho-1}{\rho}\right) P_{ijk} + \left(\frac{1}{\rho}-\frac{1}{\eta}\right) P_{ijk}\left(\frac{q_{ijk}^{1-1/\rho}}{\sum_{l=1}^{K} q_{ijl}^{1-1/\rho}}\right) = MC_{ijk}.$$

Note that

$$P_{ijk} q_{ijk} = P_i(c_i^{1/\eta})(q_{ijk}^{1-1/\rho})(y_{ij}^{1/\rho-1/\eta})$$
$$\sum_{l=1}^{K} P_{ijl} q_{ijl} = P_i(c_i^{1/\eta})(y_{ij}^{1/\rho-1/\eta}) \sum_{l=1}^{K} q_{ijl}^{1-1/\rho},$$

so $\frac{q_{ijk}^{1-1/\rho}}{\sum_{l=1}^{K} q_{ijl}^{1-1/\rho}} = \frac{P_{ijk}q_{ijk}}{\sum_{l=1}^{K} P_{ijl}q_{ijl}} \equiv s_{ijk}$, which is the market share of firm k in its sector. With this, the first order condition can be simplified to

$$\left(\frac{\rho-1}{\rho}\right) P_{ijk} + \left(\frac{1}{\rho} - \frac{1}{\eta}\right) P_{ijk}s_{ijk} = MC_{ijk} \quad \Rightarrow \quad P_{ijk} = \left[\frac{\eta(\rho-1)+s(\eta-\rho)}{\eta\rho}\right]^{-1} MC_{ijk},$$

which can be shown is the same as equation (15) in the paper.

2 An alternative derivation from Heise et.al (2022)

Consider an economy composed by (a finitite number of) S sectors indexed by s. Production is carried out by a competitive firm combining output from a continuum of industries $k \in [0,1]$, with a technology[1]

$$Y_s = \left(\int_0^1 y_s(k)^{\frac{\sigma-1}{\sigma}} dk\right)^{\frac{\sigma}{\sigma-1}}.$$

The problem of this producer generates the following demand and price index

$$y_s(k) = \left(\frac{p_s(k)}{P_s}\right)^{-\sigma} Y_s \quad \text{and} \quad P_s = \left(\int_0^1 p_s(k)^{1-\sigma} dk\right)^{\frac{1}{1-\sigma}}.$$

Each industry is populated by finite number of firms, $N_s(k)$, which are indexed by i. (In what follows, $N_s(k) = N$ for simplicity.) The industry-specific aggregator takes the form

$$y_s(k) = \left(\sum_{i=1}^{N} y_s(k,i)^{\frac{\eta-1}{\eta}}\right)^{\frac{\eta}{\eta-1}}.$$

Therefore, the demand for variety (k,i) and the price index are

$$y_s(k,i) = p_s(k,i)^{-\eta} p_s(k)^{\eta-\sigma} P_s^{\sigma} Y_s \quad \text{and} \quad p_s(k) = \left(\sum_{i=1}^{N} p_s(k,i)^{1-\eta}\right)^{\frac{1}{1-\eta}}.$$

Each firm produces with a Cobb-Douglas technology $y = Al^{\alpha}k^{1-\alpha}$, such that the marginal cost is

$$c_s(k,i) = \frac{1}{A} w^{\alpha} r^{1-\alpha}.$$

[1]"The missing inflation puzzle: The role of the wage-price pass-through" by Sebastian Heise, Fatih Karahan and Aysegul Sahin. See paper at https://www.nber.org/system/files/working_papers/w27663/w27663.pdf.

Market structure and demand elasticity. The key assumption, as in Atkenson and Burstein (2008), is that varieties are more substitutible across firms in the same industry than across industries, $\eta > \sigma > 1$. Firms compete under Bertrand competition, taking as given the prices chosen by other firms when setting their price, and taking as given input costs. Since there is only a finite number of firms, each firm takes into account the effect of its price setting on the price index $p_s(k)$.
Define the effective demand elasticity for a firm as

$$\mathcal{E}_s(k,i) \equiv -\frac{d\log y_s(k,i)}{d\log p_s(k,i)} = \eta - (\eta-\sigma)\frac{d\log p_s(k)}{d\log p_s(k,i)}.$$

From the definition of the price index

$$\log p_s(k) = \frac{1}{1-\eta}\log\left(\sum_{i=1}^{N}\exp[(1-\eta)\log p_s(k,i)]\right) \quad\Rightarrow\quad \frac{d\log p_s(k)}{d\log p_s(k,i)} = \frac{p_s(k,i)^{1-\eta}}{\sum_{i'=1}^{N} p_s(k,i')^{1-\eta}}.$$

On the other hand, define the market share of a firm as $\varphi_s(k,i) = p_s(k,i)y_s(k,i)/(\sum_{i'=1} p_s(k,i')y_s(k,i'))$. Replacing the effective demand in the latter expression, we have

$$\varphi_s(k,i) = \frac{p_s(k,i)^{1-\eta}}{\sum_{i'=1}^{N} p_s(k,i')^{1-\eta}}.$$

Therefore, the effective demand elasticity can be written as

$$\mathcal{E}_s(k,i) = \eta - (\eta-\sigma)\varphi_s(k,i) = \eta[1-\varphi_s(k,i)] + \sigma\varphi_s(k,i).$$

Thus, the firm's demand elasticity is a weighted average of the within-industry and across-industry elasticities of substitution.

Price setting. The firm's profix maximization problem is

$$\begin{aligned}\max_{p_s(k,i)}[p_s(k,i)-c_s(k,i)]y_s(k,i) &= \max_{p_s(k,i)}[p_s(k,i)-c_s(k,i)]p_s(k,i)^{-\eta}p_s(k)^{\eta-\sigma}P_s^{\sigma}Y_s \\ &= \max_{p_s(k,i)}[p_s(k,i)^{1-\eta}p_s(k)^{\eta-\sigma} - c_s(k,i)p_s(k,i)^{-\eta}p_s(k)^{\eta-\sigma}]P_s^{\sigma}Y_s\end{aligned}$$

The first-order condition reads as

$$\left[(1-\eta)p_s(k,i)^{-\eta} + c_s(k,i)\eta p_s(k,i)^{-\eta-1}\right]p_s(k)^{\eta-\sigma} + (\eta-\sigma)p_s(k,i)^{-\eta}p_s(k)^{\eta-\sigma-1}\frac{dp_s(k)}{dp_s(k,i)}(p_s(k,i)-c_s(k,i)) = 0.$$

Note, however, that $\frac{dp_s(k)}{dp_s(k,i)} = \left(\frac{p_s(k)}{p_s(k,i)}\right)^{\eta}$ and $\varphi_s(k,i) = \left(\frac{p_s(k,i)}{p_s(k)}\right)^{1-\eta}$. Replacing in the first-order condition and re-ordering

$$p_s(k,i) = \frac{\eta-(\eta-\sigma)\varphi_s(k,i)}{(\eta-1)-(\eta-\sigma)\varphi_s(k,i)}c_s(k,i) = \frac{\mathcal{E}_s(k,i)}{\mathcal{E}_s(k,i)-1}c_s(k,i) = \mathcal{M}_s(k,i)c_s(k,i).$$