Small-Sample Bias Correction in Dynamic Term Structure Models

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November, 2026

Our goal is to understand how correcting the small-sample bias of a persistent VAR changes the decomposition of a bond yield into expected future short rates and a term premium. The essential distinction is between the dynamics used to *forecast rates* and those used to *price bonds*.

1 A canonical Gaussian affine model

Let $X_t \in \mathbb{R}^K$ be the vector of pricing factors. Under the physical measure $\mathbb{P}$, assume

$$X_{t+1} = c^P + \Phi^P X_t + \Sigma \varepsilon^P_{t+1}, \qquad \varepsilon^P_{t+1} \mid \mathcal{F}_t \sim N(0, I), \tag{1}$$

$$r_t = \delta_0 + \delta_1' X_t, \tag{2}$$

where r_t is the one-period continuously compounded short rate. A Gaussian VAR(p) can be written in this form by stacking lags in X_t.

Consider an affine price of risk, $\lambda_t = \lambda_0 + \lambda_1 X_t$, and the stochastic discount factor

$$M_{t+1} = \exp\left(-r_t - \tfrac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon^P_{t+1}\right). \tag{3}$$

The resulting risk-neutral dynamics are $X_{t+1} = c^Q + \Phi^Q X_t + \Sigma \varepsilon^Q_{t+1}$, with

$$c^Q = c^P - \Sigma\lambda_0, \qquad \Phi^Q = \Phi^P - \Sigma\lambda_1. \tag{4}$$

The price of an n-period zero-coupon bond is exponential-affine, $P^{(n)}_t = \exp(A_n + B_n' X_t)$. No arbitrage gives the recursions

$$\begin{aligned} A_{n+1} &= A_n + B_n' c^Q + \tfrac{1}{2} B_n' \Sigma\Sigma' B_n - \delta_0, \\ B_{n+1} &= (\Phi^Q)' B_n - \delta_1, \qquad A_0 = B_0 = 0, \end{aligned} \tag{5}$$

so the model-implied yield is $y^{(n)}_t = -(A_n + B_n' X_t)/n$. Hence Φ^Q *governs bond prices, whereas* Φ^P *governs forecasts of short rates*. The two matrices need not change together when the physical VAR is bias-corrected (see also Adrian, Crump, and Moench, 2013).

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We define the expectations component as the arithmetic average of future short rates under $\mathbb{P}$:

$$\mathrm{RN}_t^{(n)} \equiv \frac{1}{n}\sum_{j=0}^{n-1}\mathbb{E}_t^P(r_{t+j}). \tag{6}$$

If Φ^P is stable, let $\mu^P = (I-\Phi^P)^{-1}c^P$ and define $H_n(\Phi) = n^{-1}\sum_{j=0}^{n-1}\Phi^j$. Iterating Eq. (1) gives

$$\begin{aligned}\mathbb{E}_t^P X_{t+j} &= \mu^P + (\Phi^P)^j(X_t - \mu^P),\\ \mathrm{RN}_t^{(n)} &= \delta_0 + \delta_1'\left[\mu^P + H_n(\Phi^P)(X_t-\mu^P)\right].\end{aligned} \tag{7}$$

We define the term premium as the residual, $\mathrm{TP}_t^{(n)} \equiv y_{t,\mathrm{obs}}^{(n)} - \mathrm{RN}_t^{(n)}$.[1]

For detailed derivations of the change from $\mathbb{P}$ to $\mathbb{Q}$, the bond-pricing recursion, and the interpretation of the "risk-neutral" label, see Appendix A.

2 What does the bias correction change?

In a persistent VAR, OLS estimates of the autoregressive coefficients can be biased in small samples. In particular, persistence is often underestimated, although neither the sign of every coefficient bias nor the direction of every eigenvalue correction is guaranteed in a multivariate VAR. Bauer, Rudebusch, and Wu (2012) use a simulation-based bias correction: the physical parameters are adjusted so that, when data are simulated from the proposed model, the resulting OLS estimates reproduce those found in the observed sample. We want to understand what changes in the yield decomposition when the estimated physical dynamics are corrected.

Write $i = 0$ for the original estimate and $i = 1$ for the corrected estimate. Both use the same observed factors X_t, the same short-rate equation $r_t = \delta_0 + \delta_1' X_t$, and the same maturity n. Their physical dynamics are

$$X_{t+1} = c_i^P + \Phi_i^P X_t + \Sigma\varepsilon_{t+1}^P, \qquad i \in \{0,1\}. \tag{8}$$

We hold Σ fixed to isolate the change in the VAR coefficients. The algebra below does not otherwise require the corrected and uncorrected models to have the same pricing parameters under $\mathbb{Q}$.

From the VAR to expected short rates. Iterating Eq. (8) and taking conditional expectations gives

$$\begin{aligned}\mathbb{E}_t^P[X_{t+1}] &= c_i^P + \Phi_i^P X_t,\\ \mathbb{E}_t^P[X_{t+2}] &= c_i^P + \Phi_i^P c_i^P + (\Phi_i^P)^2 X_t,\\ \mathbb{E}_t^P[X_{t+j}] &= \sum_{k=0}^{j-1}(\Phi_i^P)^k c_i^P + (\Phi_i^P)^j X_t.\end{aligned}$$

[1] The arithmetic expected-rate component is not identical in levels to a hypothetical yield priced under $\mathbb{P}$, because the latter contains a Gaussian convexity adjustment. Under constant model parameters that adjustment is time-invariant, so it cancels from within-model changes.

Assume that Φ_i^P is stable, and define its implied long-run mean as

$$\mu_i = (I - \Phi_i^P)^{-1} c_i^P, \qquad c_i^P = (I - \Phi_i^P)\mu_i. \tag{9}$$

Since $\sum_{k=0}^{j-1}(\Phi_i^P)^k(I - \Phi_i^P) = I - (\Phi_i^P)^j$, we obtain

$$\mathbb{E}_t^P[X_{t+j}] = \mu_i + (\Phi_i^P)^j(X_t - \mu_i). \tag{10}$$

Substituting this expression into the definition of RN in Eq. (6) yields

$$\begin{aligned} \mathrm{RN}_{i,t}^{(n)} &= \frac{1}{n}\sum_{j=0}^{n-1}\left\{\delta_0 + \delta_1'\left[\mu_i + (\Phi_i^P)^j(X_t - \mu_i)\right]\right\} \\ &= \delta_0 + \delta_1'\left[\mu_i + H_i(X_t - \mu_i)\right], \qquad H_i \equiv \frac{1}{n}\sum_{j=0}^{n-1}(\Phi_i^P)^j. \end{aligned} \tag{11}$$

For each estimator, RN is an affine function of the current factors:

$$\mathrm{RN}_{i,t}^{(n)} = a_i + b_i' X_t, \qquad a_i = \delta_0 + \delta_1'(I - H_i)\mu_i, \qquad b_i = H_i'\delta_1. \tag{12}$$

The vector b_i measures the response of expected short rates to movements in the factors. The intercept a_i determines the remaining part of their level.

Two channels through which the correction changes the level. Subtracting Eq. (11) across estimators,

$$\begin{aligned} \mathrm{RN}_{1,t}^{(n)} - \mathrm{RN}_{0,t}^{(n)} &= \delta_1'\left\{H_1 X_t + (I - H_1)\mu_1 - H_0 X_t - (I - H_0)\mu_0\right\} \\ &= \delta_1'\left\{(H_1 - H_0)(X_t - \mu_0) + (I - H_1)(\mu_1 - \mu_0)\right\}. \end{aligned}$$

Consequently,

$$\mathrm{RN}_{1,t}^{(n)} - \mathrm{RN}_{0,t}^{(n)} = \underbrace{\delta_1'(H_1 - H_0)(X_t - \mu_0)}_{\text{persistence / propagation}} + \underbrace{\delta_1'(I - H_1)(\mu_1 - \mu_0)}_{\text{long-run anchor}}. \tag{13}$$

This decomposition is exact. The first term changes how the current deviation from the original long-run mean propagates into future rates. The second changes the long-run mean itself. Thus, a correction to persistence does *not* necessarily increase the level of expected future rates.

For a stable VAR, the limiting expected short rate is

$$\lim_{j\to\infty} \mathbb{E}_t^P[r_{t+j}] = \delta_0 + \delta_1'\mu_i. \tag{14}$$

This is different from the average expectation over a finite maturity n, which depends on both μ_i and H_i. The distinction matters most when the estimated VAR is very persistent.

Why the treatment of the intercept matters. There are two useful ways to compare the corrected and uncorrected estimates.

First, impose a common long-run mean m by setting $c_i^P = (I - \Phi_i^P)m$. Then $\mu_0 = \mu_1 = m$, and Eq. (13) reduces to

$$\mathrm{RN}_{1,t}^{(n)} - \mathrm{RN}_{0,t}^{(n)} = \delta_1'(H_1 - H_0)(X_t - m). \tag{15}$$

This isolates the propagation channel.

Second, allow the intercept to be re-estimated when the autoregressive matrix changes. To see what this does, suppose we have T observed transitions, $X_0, \ldots, X_T$. For a candidate matrix Φ, the least-squares intercept minimizes

$$Q(c; \Phi) = \sum_{t=0}^{T-1} \|X_{t+1} - c - \Phi X_t\|^2.$$

Differentiating with respect to c and setting the result equal to zero,

$$0 = \sum_{t=0}^{T-1} (X_{t+1} - c - \Phi X_t),$$
$$Tc = \sum_{t=0}^{T-1} X_{t+1} - \Phi \sum_{t=0}^{T-1} X_t.$$

Define $\bar{X}_+ = T^{-1} \sum_{t=0}^{T-1} X_{t+1}$ and $\bar{X}_- = T^{-1} \sum_{t=0}^{T-1} X_t$. The implied intercept and long-run mean are

$$c^*(\Phi) = \bar{X}_+ - \Phi \bar{X}_-,$$
$$\mu^*(\Phi) = (I - \Phi)^{-1} c^*(\Phi) = \bar{X}_- + (I - \Phi)^{-1} \frac{X_T - X_0}{T}. \tag{16}$$

The last equality uses $\bar{X}_+ - \bar{X}_- = (X_T - X_0)/T$. Therefore, when Φ has a root close to one, the matrix $(I - \Phi)^{-1}$ can make the implied long-run mean highly sensitive to the endpoints of the sample. Simply modifying Φ while leaving c unchanged represents a third convention and generally changes μ as well. The choice must be made explicit.

A one-factor example. Consider

$$x_{t+1} = (1 - \rho)m + \rho x_t + \varepsilon_{t+1}, \qquad r_t = \delta_0 + d x_t, \qquad 0 < \rho < 1.$$

From Eq. (11),

$$\mathrm{RN}_t^{(n)} = \delta_0 + d\,[m + h_n(\rho)(x_t - m)]\,,$$
$$h_n(\rho) \equiv \frac{1}{n} \sum_{j=0}^{n-1} \rho^j = \frac{1 - \rho^n}{n(1 - \rho)}. \tag{17}$$

Keeping m fixed, differentiation gives

$$\left.\frac{\partial \mathrm{RN}_t^{(n)}}{\partial \rho}\right|_m = d(x_t - m) h_n'(\rho), \qquad h_n'(\rho) = \frac{1}{n} \sum_{j=1}^{n-1} j \rho^{j-1} > 0 \quad (n > 1). \tag{18}$$

For $d > 0$, higher persistence raises RN if $x_t > m$, lowers it if $x_t < m$, and leaves it unchanged when $x_t = m$. If the correction also changes m, the total change instead obeys the two-channel formula in Eq. (13). In particular, there is no general sign prediction for the level of RN.

Relative contribution to the yield level. For the same observed yield $y^{(n)}_{t,\text{obs}}$, define $\text{TP}^{(n)}_{i,t} = y^{(n)}_{t,\text{obs}} - \text{RN}^{(n)}_{i,t}$. Therefore,

$$\text{TP}^{(n)}_{1,t} - \text{TP}^{(n)}_{0,t} = -\left[\text{RN}^{(n)}_{1,t} - \text{RN}^{(n)}_{0,t}\right]. \tag{19}$$

When the observed yield is nonzero, one can report the level shares

$$s^{\text{level}}_{\text{RN},i,t} = \frac{\text{RN}^{(n)}_{i,t}}{y^{(n)}_{t,\text{obs}}}, \qquad s^{\text{level}}_{\text{TP},i,t} = \frac{\text{TP}^{(n)}_{i,t}}{y^{(n)}_{t,\text{obs}}} = 1 - s^{\text{level}}_{\text{RN},i,t}. \tag{20}$$

The correction changes these shares by

$$s^{\text{level}}_{\text{RN},1,t} - s^{\text{level}}_{\text{RN},0,t} = \frac{\text{RN}^{(n)}_{1,t} - \text{RN}^{(n)}_{0,t}}{y^{(n)}_{t,\text{obs}}}, \qquad \Delta s^{\text{level}}_{\text{TP},t} = -\Delta s^{\text{level}}_{\text{RN},t}. \tag{21}$$

These shares are accounting ratios: they may be negative or exceed one if the components have opposite signs, and are unstable when the observed yield is near zero.

3 Implications for yield changes and the term premium

The previous section concerns the *level* of expected short rates. We now examine changes between dates, the sensitivity to factor movements, and the variance of the two components. Throughout this section, each parameter estimate is held fixed when evaluating different dates.

Changes in expected rates and in the term premium. Let s and t be two observation dates and put $v = X_t - X_s$. By Eq. (12),

$$\begin{aligned} R_i \equiv \text{RN}^{(n)}_{i,t} - \text{RN}^{(n)}_{i,s} &= (a_i + b_i' X_t) - (a_i + b_i' X_s) \\ &= b_i' v = \delta_1' H_i v. \end{aligned} \tag{22}$$

Thus, the intercept and long-run mean cancel *exactly* from within-model changes. The bias correction changes only the loading on the common factor movement:

$$D \equiv R_1 - R_0 = (b_1 - b_0)' v = \delta_1'(H_1 - H_0)v. \tag{23}$$

Let $Y \equiv y^{(n)}_{t,\text{obs}} - y^{(n)}_{s,\text{obs}}$ be the common observed yield change, and set $P_i \equiv \text{TP}^{(n)}_{i,t} - \text{TP}^{(n)}_{i,s}$. Since $\text{TP}_i = y_{\text{obs}} - \text{RN}_i$,

$$Y = R_i + P_i, \qquad P_i = Y - R_i, \qquad R_1 = R_0 + D, \qquad P_1 = P_0 - D. \tag{24}$$

A correction can therefore change the explanation of a yield movement without changing the yield movement itself.

For $Y \neq 0$, the relative contributions to an individual yield change are

$$s^{\text{change}}_{\text{RN},i} = \frac{R_i}{Y}, \qquad s^{\text{change}}_{\text{TP},i} = \frac{P_i}{Y} = 1 - s^{\text{change}}_{\text{RN},i}. \tag{25}$$

These contributions may also be negative or exceed 100%. They become uninformative when the observed yield change is very small.

Why persistence changes the sensitivity to factor movements. The $K \times 1$ loading in Eq. (22) is

$$b_i = H_n(\Phi_i^P)'\delta_1, \qquad H_n(\Phi) = \frac{1}{n}(I + \Phi + \cdots + \Phi^{n-1}). \tag{26}$$

For arbitrary square matrices A and B (which need not commute),

$$A^j - B^j = \sum_{k=0}^{j-1} A^{j-1-k}(A - B)B^k.$$

To verify this identity, expand each summand as $A^{j-k}B^k - A^{j-1-k}B^{k+1}$: all intermediate terms cancel. Taking $A = \Phi_1^P$ and $B = \Phi_0^P$ gives

$$H_1 - H_0 = \frac{1}{n}\sum_{j=1}^{n-1}\sum_{k=0}^{j-1}(\Phi_1^P)^{j-1-k}(\Phi_1^P - \Phi_0^P)(\Phi_0^P)^k. \tag{27}$$

This is the exact propagation of the VAR correction into the sensitivity of expected rates. It depends on the complete matrix and on the direction of the factor movement v; it is not generally determined by a single eigenvalue.

In the one-factor example in Eq. (17),

$$R_i = d\, h_n(\rho_i)\Delta x, \qquad R_1 - R_0 = d\,[h_n(\rho_1) - h_n(\rho_0)]\Delta x. \tag{28}$$

If $0 < \rho_0 < \rho_1 < 1$ and $n > 1$, then $h_n(\rho_1) > h_n(\rho_0)$. Thus $|R_1| > |R_0|$ for a nonzero factor change (provided $d \neq 0$). In this special case, correcting persistence upward makes the expected-rate component more sensitive to the same movement in the factor. The analogous conclusion is not guaranteed for a multivariate VAR.

The decomposition of yield volatility. For a fixed maturity and a common set of observation dates, take variances and covariances of the yield changes in Eq. (24). By bilinearity of covariance,

$$\begin{aligned}\mathrm{Var}(Y) &= \mathrm{Var}(R_i + P_i) \\ &= \mathrm{Var}(R_i) + \mathrm{Var}(P_i) + 2\,\mathrm{Cov}(R_i, P_i).\end{aligned}$$

Therefore,

$$\mathrm{Var}(Y) = \mathrm{Var}(R_i) + \mathrm{Var}(P_i) + 2\,\mathrm{Cov}(R_i, P_i). \tag{29}$$

The observed yield variance is the same before and after correction. The three terms on the right-hand side, however, need not be the same.

Let $\Omega \equiv \mathrm{Var}(v)$ denote the covariance matrix of the *same* factor changes used to compute R_i. Since $R_i = b_i'v$,

$$\mathrm{Var}(R_i) = \mathrm{Var}(b_i'v) = b_i'\Omega b_i. \tag{30}$$

This expression makes clear why higher estimated persistence need not always increase the variance of R_i: the correction changes the entire vector b_i, and the resulting effect depends on Ω. For an arbitrary observed yield, also define the vector $k \equiv \mathrm{Cov}(v, Y)$. We then have

$$\begin{aligned}
\mathrm{Cov}(R_i, Y) &= b_i' k, \\
\mathrm{Cov}(R_i, P_i) &= \mathrm{Cov}(R_i, Y - R_i) = b_i' k - b_i' \Omega b_i, \qquad (31) \\
\mathrm{Var}(P_i) &= \mathrm{Var}(Y - R_i) = \mathrm{Var}(Y) + b_i' \Omega b_i - 2 b_i' k. \qquad (32)
\end{aligned}$$

These equalities are valid even when the observed yield is not exactly fitted by the affine model.

The exactly fitted affine case. From the pricing equation in Eq. (5), the model-implied yield can be written

$$y_t^{(n)} = -\frac{A_n}{n} + q_n' X_t, \qquad q_n \equiv -\frac{B_n}{n}.$$

If this yield exactly matches the observed yield, and its $\mathbb{Q}$ parameters are held fixed across the two physical estimates, then $Y = q_n' v$. In this case $k = \Omega q_n$ and

$$\begin{aligned}
\mathrm{Var}(R_i) &= b_i' \Omega b_i, \\
\mathrm{Var}(P_i) &= (q_n - b_i)' \Omega (q_n - b_i), \\
\mathrm{Cov}(R_i, P_i) &= b_i' \Omega (q_n - b_i). \qquad (33)
\end{aligned}$$

The covariance can be written as

$$\mathrm{Cov}(R_i, P_i) = \frac{1}{4} q_n' \Omega q_n - \left(b_i - \frac{1}{2} q_n\right)' \Omega \left(b_i - \frac{1}{2} q_n\right). \qquad (34)$$

Consequently, the covariance becomes negative when the quadratic distance between b_i and $q_n/2$, measured using the positive-semidefinite matrix Ω, is sufficiently large. In multiple dimensions, the direction of the loading change matters. If the observed yield contains a fitting error, use Eqs. (31) and (32) rather than the exact-fit expressions.

What changes in the variance decomposition after correction? The definitions $R_1 = R_0 + D$ and $P_1 = P_0 - D$ in Eq. (24) yield

$$\begin{aligned}
\mathrm{Var}(R_1) - \mathrm{Var}(R_0) &= 2\,\mathrm{Cov}(R_0, D) + \mathrm{Var}(D), \qquad (35) \\
\mathrm{Var}(P_1) - \mathrm{Var}(P_0) &= -2\,\mathrm{Cov}(P_0, D) + \mathrm{Var}(D), \qquad (36) \\
\mathrm{Cov}(R_1, P_1) - \mathrm{Cov}(R_0, P_0) &= \mathrm{Cov}(D, P_0 - R_0) - \mathrm{Var}(D). \qquad (37)
\end{aligned}$$

The last identity follows directly from

$$\begin{aligned}
\mathrm{Cov}(R_0 + D, P_0 - D) = \mathrm{Cov}(R_0, P_0) &- \mathrm{Cov}(R_0, D) \\
&+ \mathrm{Cov}(D, P_0) - \mathrm{Var}(D).
\end{aligned}$$

Adding Eqs. (35) and (36) and twice Eq. (37) gives zero, as required: the change in the two component variances is exactly offset by the change in their covariance. In particular, *both component variances can increase while the observed yield variance remains unchanged*.

There are two useful ways to report relative contributions to yield volatility. The direct variance accounting is

$$1 = \frac{\mathrm{Var}(R_i)}{\mathrm{Var}(Y)} + \frac{\mathrm{Var}(P_i)}{\mathrm{Var}(Y)} + \frac{2\,\mathrm{Cov}(R_i, P_i)}{\mathrm{Var}(Y)}, \qquad \mathrm{Var}(Y) > 0. \tag{38}$$

The two individual variance ratios need not add to one. If a decomposition into exactly two contributions that sum to one is desired, one may allocate the covariance symmetrically:

$$\begin{aligned} s^{\mathrm{var}}_{\mathrm{RN},i} &= \frac{\mathrm{Var}(R_i) + \mathrm{Cov}(R_i, P_i)}{\mathrm{Var}(Y)} = \frac{\mathrm{Cov}(R_i, Y)}{\mathrm{Var}(Y)}, \\ s^{\mathrm{var}}_{\mathrm{TP},i} &= \frac{\mathrm{Var}(P_i) + \mathrm{Cov}(R_i, P_i)}{\mathrm{Var}(Y)} = \frac{\mathrm{Cov}(P_i, Y)}{\mathrm{Var}(Y)}, \\ s^{\mathrm{var}}_{\mathrm{RN},i} + s^{\mathrm{var}}_{\mathrm{TP},i} &= 1. \end{aligned} \tag{39}$$

These are accounting attributions, not independent causal contributions; either share can be negative or exceed one.

A one-factor illustration of a sign reversal. Suppose $r_t = \delta_0 + dx_t$ and both the physical and risk-neutral factor dynamics have scalar autoregressive coefficients in $(0, 1)$. For an exactly fitted yield and $n > 1$, its factor loading is $q_n = d\,h_n(\rho^Q)$, while the expected-rate loading is $b_n = d\,h_n(\rho^P)$. Therefore,

$$\mathrm{Cov}(\Delta\mathrm{RN}, \Delta\mathrm{TP}) = d^2 h_n(\rho^P)\left[h_n(\rho^Q) - h_n(\rho^P)\right]\mathrm{Var}(\Delta x). \tag{40}$$

Because $h_n(\rho) > 0$ and $h_n'(\rho) > 0$ on $(0, 1)$,

$$\mathrm{sign}\left[\mathrm{Cov}(\Delta\mathrm{RN}, \Delta\mathrm{TP})\right] = \mathrm{sign}(\rho^Q - \rho^P), \tag{41}$$

provided $d \neq 0$ and $\mathrm{Var}(\Delta x) > 0$. This sign result is specific to the one-factor, exactly fitted case; it does not follow from persistence alone with several factors or pricing errors.

For example, consider monthly dynamics, $n = 120$, $d = 1$, and $\rho^Q = 0.98$, $\rho_0^P = 0.97$, $\rho_1^P = 0.99$. Then

$$h_{120}(0.97) \simeq 0.271, \qquad h_{120}(0.98) \simeq 0.380, \qquad h_{120}(0.99) \simeq 0.584.$$

For the same 100-basis-point increase in x,

	Original VAR	Corrected VAR
Observed yield change Y	+38.0 bp	+38.0 bp
Expected-rate change R_i	+27.1 bp	+58.4 bp
Term-premium change P_i	+10.9 bp	−20.4 bp

The correction increases the sensitivity of expected rates sufficiently to make the implied change in the term premium negative. If, across observation dates, $\mathrm{sd}(\Delta x) = 100$ basis points, the standard deviations

of R_i are 27.1 and 58.4 basis points, and those of P_i are 10.9 and 20.4 basis points. Yet the standard deviation of Y remains 38.0 basis points. The covariance switches from positive to negative, offsetting the larger component variances.

The main lesson is that correcting the physical VAR changes the *allocation* of the same observed yield movements across expected short rates and the term premium. Whether this improves their accuracy is a separate question: the sign of the covariance or the size of the variance shares alone cannot establish that the corrected expectations are closer to the underlying physical expectations.

4 Summary

Small-sample bias correction in Gaussian affine term structure models primarily affects the estimated dynamics of the pricing factors under the physical measure $\mathbb{P}$. Because expected future short rates depend on these dynamics, correcting the VAR changes the decomposition of observed yields into expected short rates (RN) and term premiums (TP). The main implications can be summarized as follows:

- **Persistence.** The conventional small-sample bias of a persistent VAR tends to underestimate the persistence of the factors. Bias correction typically increases the estimated persistence under $\mathbb{P}$, implying that shocks to the factors are expected to dissipate more slowly. This is not a guaranteed outcome for every coefficient or eigenvalue in a multivariate VAR.
- **Long-run expectations.** Changes in persistence and in the VAR intercept affect the unconditional mean of the factors and, consequently, the long-run expected short rate. The direction and magnitude of this effect depend on how the intercept is treated when correcting the VAR.
- **Level decomposition.** Correcting the VAR changes the estimated level of RN and, residually, TP. However, the direction is generally ambiguous: a higher estimated persistence does not necessarily imply a higher level of RN. The result depends on the current state of the factors relative to their long-run mean.
- **Yield changes.** Bias correction changes the sensitivity of RN to movements in the factors. In a scalar model with positive persistence, increasing persistence raises this sensitivity. In a multivariate VAR, the effect depends on the entire transition matrix and the direction of factor movements. Since observed yields remain unchanged, changes in RN are offset by corresponding changes in TP.
- **Volatility decomposition.** Changes in the factor sensitivities affect the variances of RN and TP and their covariance. In particular, both components can become more volatile even when the variance of observed yields remains unchanged. Bias correction can also change the sign of the covariance between RN and TP, reflecting a different allocation of yield movements between the two components.

The central implication is that correcting small-sample bias changes how the model interprets the same observed yield curve. In particular, greater estimated persistence can assign a larger fraction of yield

movements to revisions in expected future short rates, with offsetting changes in term premiums. However, these changes are not necessarily evidence of a more accurate decomposition. Bias correction targets the estimation of the physical dynamics, while the accuracy of the resulting expectations and term premiums remains a separate question.

References

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A Physical dynamics, risk-neutral pricing, and the meaning of RN

This appendix derives three relationships used in Sections 2 and 3: how the factor dynamics under $\mathbb{Q}$ follow from the physical VAR and the stochastic discount factor, why the bond-pricing recursion uses $\mathbb{Q}$, and why an expected short-rate component constructed under $\mathbb{P}$ is often called "risk-neutral."

A.1 From the physical VAR to the risk-neutral VAR

Under $\mathbb{P}$, the pricing factors and short rate satisfy Eqs. (1) and (2):

$$\begin{aligned} X_{t+1} &= c^P + \Phi^P X_t + \Sigma \varepsilon^P_{t+1}, & \varepsilon^P_{t+1} \mid \mathcal{F}_t &\sim N(0, I), \\ r_t &= \delta_0 + \delta_1' X_t. \end{aligned} \tag{42}$$

Let the market price of risk be affine, $\lambda_t = \lambda_0 + \lambda_1 X_t$, and suppose the stochastic discount factor is

$$M_{t+1} = \exp\left(-r_t - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon^P_{t+1}\right). \tag{43}$$

The factor $-\lambda_t'\varepsilon^P_{t+1}$ compensates for exposure to innovations. Define

$$L_{t+1} \equiv e^{r_t} M_{t+1} = \exp\left(-\frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon^P_{t+1}\right). \tag{44}$$

To verify that L_{t+1} is a valid conditional change-of-measure density, use the moment-generating function of a standard Gaussian vector: if $\varepsilon \sim N(0, I)$, then $\mathbb{E}[e^{a'\varepsilon}] = e^{a'a/2}$. Conditional on the time-t information, λ_t is known. Hence

$$\begin{aligned} \mathbb{E}^P_t[L_{t+1}] &= e^{-\lambda_t'\lambda_t/2}\mathbb{E}^P_t[e^{-\lambda_t'\varepsilon^P_{t+1}}] \\ &= e^{-\lambda_t'\lambda_t/2} e^{\lambda_t'\lambda_t/2} = 1. \end{aligned} \tag{45}$$

We can therefore define conditional expectations under $\mathbb{Q}$ through

$$\mathbb{E}_t^Q[Z_{t+1}] \equiv \mathbb{E}_t^P[L_{t+1}Z_{t+1}] \tag{46}$$

for any integrable future payoff Z_{t+1}. To see how this affects the innovation distribution, multiply the standard-normal conditional density under $\mathbb{P}$ by L_{t+1}. Ignoring the normalizing constant, the resulting density is

$$\begin{aligned} f^Q(\varepsilon^P \mid \mathcal{F}_t) &\propto \exp\left(-\frac{1}{2}(\varepsilon^P)'\varepsilon^P - \frac{1}{2}\lambda_t'\lambda_t - \lambda_t'\varepsilon^P\right) \\ &= \exp\left[-\frac{1}{2}(\varepsilon^P + \lambda_t)'(\varepsilon^P + \lambda_t)\right]. \end{aligned}$$

Thus, under the new measure, ε_{t+1}^P has conditional mean $-\lambda_t$ and covariance I. If we define the new innovation as

$$\varepsilon_{t+1}^Q \equiv \varepsilon_{t+1}^P + \lambda_t, \qquad \varepsilon_{t+1}^Q \mid \mathcal{F}_t \overset{Q}{\sim} N(0, I), \tag{47}$$

then $\varepsilon_{t+1}^P = \varepsilon_{t+1}^Q - \lambda_t$. Substituting into Eq. (42),

$$\begin{aligned} X_{t+1} &= c^P + \Phi^P X_t + \Sigma(\varepsilon_{t+1}^Q - \lambda_t) \\ &= c^P + \Phi^P X_t + \Sigma\varepsilon_{t+1}^Q - \Sigma(\lambda_0 + \lambda_1 X_t) \\ &= (c^P - \Sigma\lambda_0) + (\Phi^P - \Sigma\lambda_1)X_t + \Sigma\varepsilon_{t+1}^Q. \end{aligned}$$

This proves the risk-neutral VAR:

$$\begin{aligned} X_{t+1} &= c^Q + \Phi^Q X_t + \Sigma\varepsilon_{t+1}^Q, \\ c^Q &= c^P - \Sigma\lambda_0, \qquad \Phi^Q = \Phi^P - \Sigma\lambda_1. \end{aligned} \tag{48}$$

The covariance matrix of the innovations is unchanged by this Gaussian change of measure, but their conditional drift is shifted by the market prices of risk. Under $\mathbb{P}$, X_t follows the actual transition law. Under $\mathbb{Q}$, it follows the probability law used for valuation.

The relationship also clarifies what it means to correct Φ^P while keeping pricing dynamics fixed. If Σ is invertible, then

$$\lambda_{1,i} = \Sigma^{-1}(\Phi_i^P - \Phi^Q), \qquad \lambda_{0,i} = \Sigma^{-1}(c_i^P - c^Q). \tag{49}$$

Thus, changing physical dynamics need not change Φ^Q or c^Q: the implied prices of risk change instead. If Σ is not invertible, this adjustment is feasible only when the desired differences belong to the column space of Σ. The distinction is useful when comparing expectations while holding the fitted yield curve fixed.

A.2 Why bond prices depend on the dynamics under $\mathbb{Q}$

No arbitrage implies the one-period pricing equation

$$P_t^{(n+1)} = \mathbb{E}_t^P[M_{t+1}P_{t+1}^{(n)}]. \tag{50}$$

By Eqs. (44) and (46), $M_{t+1} = e^{-r_t} L_{t+1}$, so

$$\begin{aligned} P_t^{(n+1)} &= e^{-r_t} \mathbb{E}_t^P [L_{t+1} P_{t+1}^{(n)}] \\ &= e^{-r_t} \mathbb{E}_t^Q [P_{t+1}^{(n)}]. \end{aligned} \tag{51}$$

This is not a different pricing equation: it is the same equation written using a different probability measure. Under $\mathbb{P}$, the stochastic discount factor adjusts for risk. Under $\mathbb{Q}$, the risk adjustment is already contained in the modified transition probabilities, so discounting takes place at the one-period risk-free rate.

To derive the affine recursions, conjecture

$$P_t^{(n)} = \exp(A_n + B_n' X_t), \qquad A_0 = 0, \quad B_0 = 0. \tag{52}$$

Substituting Eqs. (48) and (52) into Eq. (51),

$$\begin{aligned} P_t^{(n+1)} &= e^{-r_t} \mathbb{E}_t^Q \left[\exp\left(A_n + B_n' X_{t+1} \right) \right] \\ &= e^{-r_t} \exp\left[A_n + B_n' (c^Q + \Phi^Q X_t) \right] \mathbb{E}_t^Q \left[\exp(B_n' \Sigma \varepsilon_{t+1}^Q) \right]. \end{aligned}$$

The scalar $B_n' \Sigma \varepsilon_{t+1}^Q$ is normal with mean zero and variance $B_n' \Sigma \Sigma' B_n$. Using the Gaussian moment-generating function once more,

$$\mathbb{E}_t^Q \left[\exp(B_n' \Sigma \varepsilon_{t+1}^Q) \right] = \exp\left(\frac{1}{2} B_n' \Sigma \Sigma' B_n \right).$$

Therefore,

$$\begin{aligned} \log P_t^{(n+1)} &= -r_t + A_n + B_n' c^Q + B_n' \Phi^Q X_t + \frac{1}{2} B_n' \Sigma \Sigma' B_n \\ &= \underbrace{\left[A_n + B_n' c^Q + \frac{1}{2} B_n' \Sigma \Sigma' B_n - \delta_0 \right]}_{A_{n+1}} \\ &\quad + \underbrace{\left[(\Phi^Q)' B_n - \delta_1 \right]'}_{B_{n+1}'} X_t. \end{aligned}$$

Matching the constant and factor terms gives precisely Eq. (5):

$$\begin{aligned} A_{n+1} &= A_n + B_n' c^Q + \frac{1}{2} B_n' \Sigma \Sigma' B_n - \delta_0, \\ B_{n+1} &= (\Phi^Q)' B_n - \delta_1. \end{aligned} \tag{53}$$

The resulting yield is

$$y_t^{(n)} = -\frac{\log P_t^{(n)}}{n} = -\frac{A_n}{n} - \frac{B_n'}{n} X_t. \tag{54}$$

The $\mathbb{Q}$ parameters determine the affine coefficients of the *priced* bond. This does not mean that actual future short rates follow $\mathbb{Q}$; their forecasts remain governed by $\mathbb{P}$.

For completeness, we can verify the equivalence directly under $\mathbb{P}$. Combining Eqs. (42), (43) and (50) and the affine conjecture yields

$$P_t^{(n+1)} = \exp\left[-r_t + A_n + B_n'(c^P + \Phi^P X_t)\right] \times \mathbb{E}_t^P\left[\exp\left(-\frac{1}{2}\lambda_t'\lambda_t + (B_n'\Sigma - \lambda_t')\varepsilon_{t+1}^P\right)\right].$$

The log of the last expectation is

$$-\frac{1}{2}\lambda_t'\lambda_t + \frac{1}{2}(\Sigma' B_n - \lambda_t)'(\Sigma' B_n - \lambda_t) = \frac{1}{2}B_n'\Sigma\Sigma' B_n - B_n'\Sigma\lambda_t.$$

Consequently,

$$\begin{aligned}\log P_t^{(n+1)} &= -r_t + A_n + B_n'(c^P + \Phi^P X_t) + \frac{1}{2}B_n'\Sigma\Sigma' B_n - B_n'\Sigma\lambda_t \\ &= -r_t + A_n + B_n'(c^Q + \Phi^Q X_t) + \frac{1}{2}B_n'\Sigma\Sigma' B_n.\end{aligned}$$

The same recursion follows. Using the physical VAR without the price-of-risk terms would generally lead to a different price.

A.3 Expected rates under $\mathbb{P}$, rates under $\mathbb{Q}$, and a risk-neutral yield

To verify that the one-period pricing recursion gives the usual discount-factor formula, start with $P_t^{(1)} = e^{-r_t}$. If, for maturity n, $P_{t+1}^{(n)} = \mathbb{E}_{t+1}^Q[\exp(-\sum_{j=0}^{n-1} r_{t+1+j})]$, then the law of iterated expectations and Eq. (51) imply

$$\begin{aligned}P_t^{(n+1)} &= e^{-r_t}\mathbb{E}_t^Q[P_{t+1}^{(n)}] \\ &= \mathbb{E}_t^Q\left[\exp\left(-r_t - \sum_{j=0}^{n-1} r_{t+1+j}\right)\right].\end{aligned}$$

The formula follows by induction for every n.

The terminology is easiest to understand by separating three objects. Define the cumulative short rate over the life of the bond as

$$S_{t,n} = \sum_{j=0}^{n-1} r_{t+j}. \tag{55}$$

First, the expected short-rate component used in the main note is the physical average

$$\mathrm{RN}_t^{(n)} \equiv \frac{1}{n}\mathbb{E}_t^P[S_{t,n}] = \frac{1}{n}\sum_{j=0}^{n-1}\mathbb{E}_t^P[r_{t+j}]. \tag{56}$$

It is a forecast under the actual probability law. The symbol RN is retained to match the convention in the main note, but it is important to remember that this definition uses $\mathbb{P}$.

Second, consider the average short rate under the actual pricing measure,

$$\bar{r}_t^{Q,(n)} \equiv \frac{1}{n}\mathbb{E}_t^Q[S_{t,n}]. \tag{57}$$

In general, $\bar{r}_t^{Q,(n)} \neq \mathrm{RN}_t^{(n)}$, because $\mathbb{Q}$ changes the conditional drift of the state variables. For example, if Φ^Q is stable, let $\mu^Q = (I - \Phi^Q)^{-1} c^Q$. Repeating the physical expectation calculation under $\mathbb{Q}$ gives

$$\bar{r}_t^{Q,(n)} = \delta_0 + \delta_1' \left[\mu^Q + H_n(\Phi^Q)(X_t - \mu^Q) \right]. \tag{58}$$

Thus, the two expected-rate averages use the same short-rate equation but different transition matrices and long-run means.

Third, consider the yield obtained in a *hypothetical economy with no prices of risk*. In that economy, the pricing measure equals the physical measure and the no-arbitrage price is

$$P_t^{(n),0} \equiv \mathbb{E}_t^P[e^{-S_{t,n}}], \qquad y_t^{(n),0} \equiv -\frac{1}{n} \log P_t^{(n),0}. \tag{59}$$

This is often called a *risk-neutral yield*. It is evaluated using $\mathbb{P}$ because we have set prices of risk to zero, making the hypothetical pricing measure coincide with $\mathbb{P}$. It should not be confused with the average $\bar{r}_t^{Q,(n)}$ under the actual pricing measure.

Under the Gaussian affine model, $S_{t,n}$ is conditionally normal. For any conditionally normal random variable S,

$$\mathbb{E}_t[e^{-S}] = \exp\left(-\mathbb{E}_t[S] + \frac{1}{2} \operatorname{Var}_t(S) \right).$$

Applying this first under the actual pricing measure,

$$\begin{aligned} P_t^{(n)} &= \mathbb{E}_t^Q[e^{-S_{t,n}}], \\ y_t^{(n)} &= \frac{1}{n} \mathbb{E}_t^Q[S_{t,n}] - \frac{1}{2n} \operatorname{Var}_t^Q(S_{t,n}). \end{aligned} \tag{60}$$

Applying the same identity under $\mathbb{P}$ to the hypothetical no-risk-price yield,

$$\begin{aligned} y_t^{(n),0} &= \frac{1}{n} \mathbb{E}_t^P[S_{t,n}] - \frac{1}{2n} \operatorname{Var}_t^P(S_{t,n}) \\ &= \mathrm{RN}_t^{(n)} - \frac{1}{2n} \operatorname{Var}_t^P(S_{t,n}). \end{aligned} \tag{61}$$

The second term is the Gaussian convexity adjustment. The arithmetic average of expected short rates and the hypothetical risk-neutral yield are therefore *not identical in levels*. If the model-implied yield matches the observed yield, combining Eqs. (56) and (60) also shows that

$$\mathrm{TP}_t^{\mathrm{ER},(n)} = \underbrace{\bar{r}_t^{Q,(n)} - \mathrm{RN}_t^{(n)}}_{\text{difference in expected rates}} \underbrace{- \frac{1}{2n} \operatorname{Var}_t^Q(S_{t,n})}_{\text{convexity under } \mathbb{Q}}. \tag{62}$$

This makes explicit where the price-of-risk adjustment enters the decomposition: it changes the expected future path under $\mathbb{Q}$ relative to that under $\mathbb{P}$.

To see why this adjustment is constant through time for fixed parameters, iterate the physical VAR and collect the innovations in $S_{t,n}$. Define

$$G_{n,k}(\Phi^P) = \sum_{h=0}^{n-1-k} (\Phi^P)^h, \qquad k = 1, \ldots, n-1.$$

The time-t innovation in the cumulative short rate is

$$S_{t,n} - \mathbb{E}_t^P[S_{t,n}] = \sum_{k=1}^{n-1} \delta_1' G_{n,k}(\Phi^P)\Sigma\varepsilon_{t+k}^P.$$

Because future innovations are conditionally independent with covariance I,

$$\mathrm{Var}_t^P(S_{t,n}) = \sum_{k=1}^{n-1} \delta_1' G_{n,k}(\Phi^P)\Sigma\Sigma' G_{n,k}(\Phi^P)'\delta_1. \tag{63}$$

The right-hand side depends on the model parameters and maturity but not on X_t. Thus, within a given estimated model,

$$\Delta y_t^{(n),0} = \Delta \mathrm{RN}_t^{(n)}, \tag{64}$$

for any two observation dates. This explains why the arithmetic expected-rate component and the hypothetical risk-neutral yield have the same factor sensitivities and changes in the homoskedastic Gaussian model. However, changing Φ^P through the bias correction also generally changes Eq. (63); the convexity adjustment need not be the same *across estimators*. Therefore, it must be kept if levels of the two definitions are compared.

Finally, the choice of convention changes the name and level of the residual component:

$$\begin{aligned} \mathrm{TP}_t^{\mathrm{ER},(n)} &\equiv y_{t,\mathrm{obs}}^{(n)} - \mathrm{RN}_t^{(n)}, \\ \mathrm{TP}_t^{0,(n)} &\equiv y_{t,\mathrm{obs}}^{(n)} - y_t^{(n),0} = \mathrm{TP}_t^{\mathrm{ER},(n)} + \frac{1}{2n}\,\mathrm{Var}_t^P(S_{t,n}). \end{aligned} \tag{65}$$

Both conventions are internally consistent. For the bias-correction exercise, it is enough to choose one and keep it unchanged throughout the analysis. The main note uses the arithmetic expectation under $\mathbb{P}$ and defines the term premium as the observed-yield residual.